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A New Circle of Modes
revisiting the concept of diatonicism

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Contents

Introduction

It is no accident that, amid the multitude of scales that can be extracted from the 12-tone equal temperament (2048, according to Keith;[1] 1490, according to Zeitler),[2] one in particular has dictated the pitch material of such a large number of leading Western composers.[3]

The diatonic scale – because that is what this is all about – has been subject in the last 40 years of increasing scholar inquiry, in part thanks to the work of mathematicians such as John Clough and David Rothenberg. Inspired equally by Milton Babbitt and Allen Forte's mathematically-based approach to serial music and by the music of the Common Practice Period, these and other mathematicians, computer scientists and music theorists paved the way for the assertion in music set theory of a new field of inquiry, that of diatonic set theory.

Still, albeit in fast growth, this field is relatively small, and foreign even to some of those involved in the study of pitch-class set theory. I myself was not aware of its existence until well after having started my own explorations of the diatonic structure. And yet, once exposed to its studies, it's all too easy to marvel at the sheer quantity of properties attesting and substantiating the unique status enjoyed by the diatonic scale, not only in the realm of theory, but in the musical practice as well.

Such an overwhelming case is postulated by these theories that I feel the need sometimes to remind their promulgators of an often overlooked fact: if indeed it is the case that there seems to be a perfect symbiosis between diatonic scale and 12-tone equal temperament, that is mostly due to the fact that, contrary to what one might think, the latter is a by-product of the former. It is not out of a particularly miraculous demonstration of serendipity that the stack of intervals which we call diatonic scale sprouts out of the 12-tone universe. Quite the contrary, it is the case that, at least since the practice of musica ficta, the History of Western Music is also the story of how the 12-tone equal temperament gradually crystallized as a system with the unequivocally sole goal of best accommodating the latest whims of diatonic scale-based composition.[4]

By framing the discourse on these terms, one can better grasp the circumstances behind the beautiful and unique design of the diatonic scale. That said, however interesting a deep dive into cataloguing the properties that help us untangle its mysterious qualities might be, that is not the aim of this thesis.

It is true that some of those properties will be addressed once more, but (and here this work differs from most of what I've seen written in this field) they will be so with the sole goal of drawing attention to the curious bond between the diatonic scale and two other scales. It is my impression that the link between these three scales, at least since its first serious address in 1963 by Lajos Bárdos in his treaty[5] about the music of Zoltán Kodály (in which he coined the terms *Heptatonia Prima*, *Secunda* and *Tertia*), have earned far too little attention from theorists, as well as from composers.

Striving in some way to fill that gap, I will momentarily turn myself into a theorist to attempt – in Chapter 2 – to explain how did I came up with these three scales (which from now on will be referred to as H1, H2, and H3) – in Chapter 3 – to build an all-inclusive, consistent modal system out of these scales – in Chapter 4 – to assess what the actual limits of that system are, and what the same properties that attest to the intrinsic qualities of the diatonic scale (H1) have to say regarding the other ones. Chapter 1 serves, for the most part, as a general introduction to some concepts of music theory, the awareness of which I believe to be imperative for a good understanding of the rest of the thesis. Still, if the reader feels totally at ease with this subject, he or she might want to tackle Chapter 2 directly.

Although this thesis is best encapsulated as an enquiry on the field of diatonic set theory, I still reserve its epilogue to address what a composer should expect when approaching the previously mentioned scales not so much as cold bricks belonging to the rigidly beautiful, lofty architecture of a modal system, but as raw material to be crafted into concrete pieces of music.

Chapter 1
The symmetry of the 12-tone equal temperament

Axial vs. rotational symmetry

As someone engaged in the more practical side of composition, I had never indulged in serious studies of Music Theory before. That said, all it was needed to trigger the entire investigation behind this thesis was a bizarrely simple realization: that the Dorian mode consists of the *inverse* of itself (by inverse I mean the relationship between two scales whose interval pattern is the opposite of each other). That is, one observes the same interval pattern whether ascending or descending on the Dorian scale.

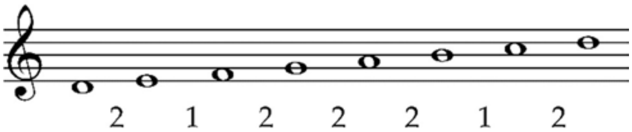


Figure 1.1

This property is called *reflective* or *axial symmetry* on the root tone, and should not be confused with *rotational symmetry* – the latter can be observed in scales whose interval pattern can be replicated through transposition.

Even though there are a few scales that combine both symmetries (the whole-tone scale being one of them), these properties are similar only in name and the harmonic challenge they pose to a composer could not be further apart – just try to imagine what Messiaen's liturgical music would sound like hadn't he embraced the rotational symmetry of his seven modes[6] and kept himself to the axial symmetry of the traditional church modes! Of course, this comparison is a bit twisted – after all, there are many other axial symmetrical modes besides the church/diatonic ones – the point being, one can expect the harmonic language extracted from an axial symmetrical scale to be of a very different nature from the one extracted from a rotational symmetrical scale.

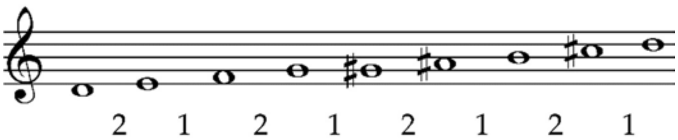


Figure 1.2

To make this clear, let us take as an example the octatonic scale / Messiaen's second mode – a rotational symmetrical mode – and the Dorian mode – an axial symmetrical one. If we look into each of the 12 transpositions of the octatonic scale, we will notice that they consist of repetitions of the same three scales starting in different roots. Employing this structure can easily lead to a very limited macroharmony[7] and, therefore, a very recognizable, "composer-friendly" harmonic language. Now, if we do the same thing for the Dorian mode and analyse its transpositions, we will get not three but twelve different pitch sets – that is, in a way, a much richer but harder to master harmonic language.

Due to the repeating nature of the interval structure of a Messiaen's mode, one does not really need to go through many of its tones for an audience to have a general good idea of the harmonic world it is witnessing.[8] But, when it comes to a diatonic mode, due to an interval structure which does not repeat by rotation, one can present most of its tones and still leave its transpositions fairly unscathed. In other words (very reductionist ones), Messiaen's modes strive to be statements, while the diatonic modes strive for development and modulation.

Diatonicism

Having said that, the core of this thesis is the study of three scales featuring *axial* symmetry, the first one being the diatonic scale.

By "diatonic" I mean the pattern of tones and semitones one can most intuitively extract from the white keys of a piano's keyboard. Depending on which of the seven keys one chooses to be the scale's root, one can find seven modes, each displaying a different interval pattern. Together they form the diatonic *family*.

Of course, diatonicism is a big concept, encapsulating different meanings in different contexts. Some expand its definition to include auxiliary scales of the Common Practice Period, such as the harmonic minor and melodic minor scales.[9]

However, for the purpose of this study, diatonicism will initially be stripped of all its adjacent scales and will refer only to the interval pattern displayed by those modes "which are the modern counterpart of the ecclesiastical modes"[10] (of course, Locrian included).

At this point I should make it clear that it is not out of neglect that those other scales mentioned above are not included in this definition – much on the contrary, one of the goals of this study is precisely to challenge that definition and, if possible, make it encompass two other scales. However, my extended conception of the term *diatonicism* will differ insofar as – contrary to all the historically informed theories which argue for it to include the variants of the minor scale – it does not rest on the context of musical practice, but on the application of a simple process of generation of scale material.

In practice, what that means is that the "diatonic" scales I will present further on are not necessarily the same as the ones someone coming from a musicological context would guess. And it is probably due to mere coincidence if in the end they happen to coincide (as one of them will in fact do). But more about that later.

A palindrome and its family

As was established before, the diatonic family features axial symmetry; effectively, what that means is that each one of the modes of one of its transpositions has *one and the same* axis of reflection (the note from which the interval pattern is the same whether ascending or descending.)

The only diatonic mode whose axis of reflection falls on the root tone is the Dorian mode (see *Figure 1.1*). A scale like this, which has the curious quality of being the inverse of itself, is called *palindromic*.[11]

Therefore, all other diatonic modes have as the axis of their reflection the root tone of the Dorian mode. Meaning that D Dorian, E Phrygian, F Lydian, G Mixolydian, A Aeolian, B Locrian and C Ionian all share the same axis of reflection – D.

And meaning also that the axis of reflection of a given family never falls on the same degree of two of its modes. For example, the axis of reflection of the diatonic family falls on the 1st degree of D Dorian; the 7th degree of E Phrygian; the 6th degree of F Lydian; the 5th degree of G Mixolydian; the 4th degree of A Aeolian; the 3rd degree of B Locrian; and the 2nd degree of C Ionian.

A mode and its inverse

When a family of modes have an axis of reflection, we can predict that each one of those modes will be the inverse of some other mode of the same family. In the case of the diatonic family:

D Dorian is the inverse of itself because the axis of reflection falls on its root tone.

E Phrygian has its root tone a major second (M2) *above* the axis of reflection, so its inverse will be the mode whose root tone is a M2 *below* the axis, that is, C Ionian.

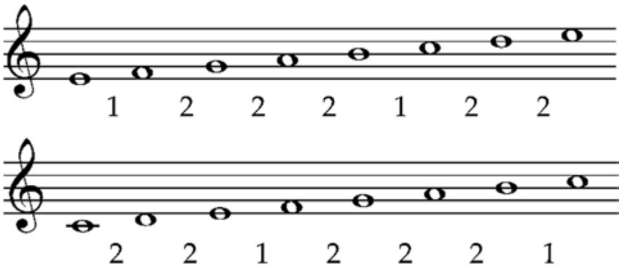


Figure 1.3

Following the same rationale, we can predict that F Lydian, which starts a minor third (m3) above D, is the inverse of B Locrian, which starts a m3 bellow D.

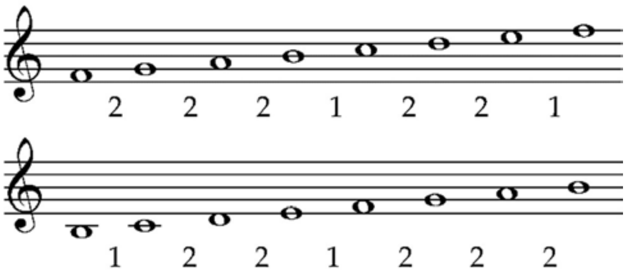


Figure 1.4

And, finally, G Mixolydian, which starts a perf ect fourth (P4) above D, is the inverse of A Aeolian, which starts a P4 bellow D.

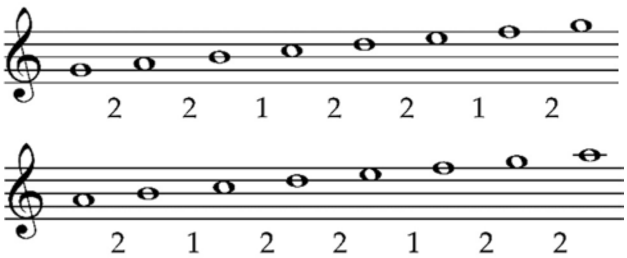


Figure 1.5

According to Vincent,[12] this curious property of the diatonic scale was first exposed by Swiss music theorist Jean Adam Serre as far back as 1753! In an appendix to his *Essais sur les Principles de l´Harmonie*, he writes (here in a free translation by me):

> *"It therefore seems to me that this natural E Mode (which can be called semi-minor Mode to define at the same time the nature of its second, and that of its third) is nothing other than the exactly reversed Major Mode. This is what we can imagine if we compare the ranges of these two Modes; we will find that one is precisely the counterpoint of the other, that is to say, that the range – mi, fa, sol, la, si, ut, re, mi – of the semi-minor Mode proceeds in ascending by exactly the same intervals by which that of C proceeds in descending, and vice versa."*[13]

Interestingly, if D is the axis of reflection of the entire diatonic family, that also makes D (together with its tritone Ab/G#) the actual axis of the entire keyboard as it has been designed. That is, no matter what combination of "white" and "black" pitches we choose, if we reflect it having D as the axis, we will get exactly the same pattern of white and black keys, only inverted. Indeed, as of 1912, Bernhard Ziehn had already pointed to that curious fact, writing that "any tone may serve as a centre, but from D only we receive relations simple and clear."[14] He then proceeded to clarify that statement, by shedding light to the obvious symmetry displayed by the key signatures containing seven flats and seven sharps – they are, of course, symmetrical to each other with D as the axis of their reflection.

Proximity between diatonic modes

Now that we determined that D Dorian is the axis of the diatonic space, one might ask which diatonic modes are closest to it – that is – which ones have the most similar interval patterns. By transposing all modes to the same transposition of D Dorian, their differences will be reflected in their key signature. Now, if we organize the resulting scales taking into account their proximity, we get a partial circle of fifths.

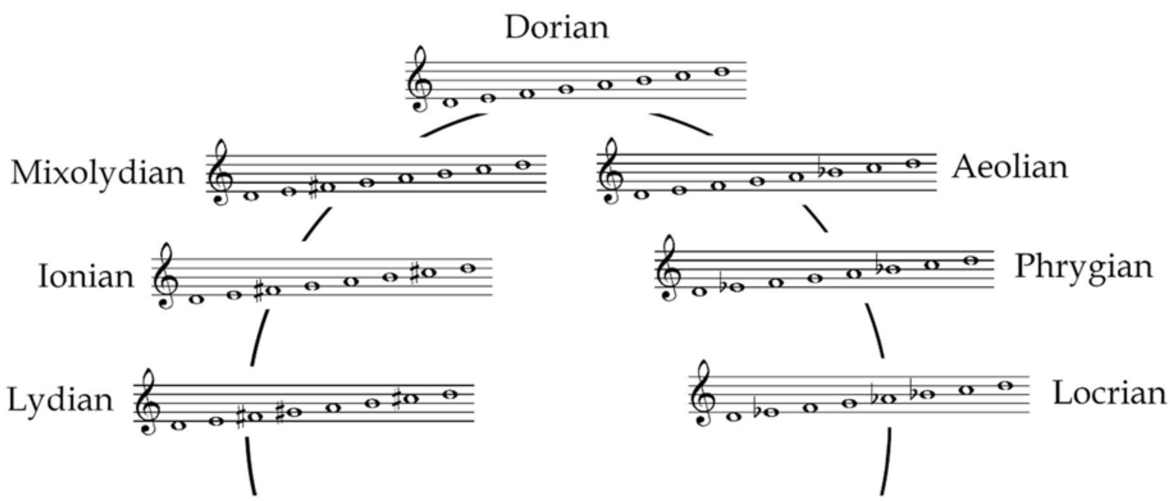


Figure 1.6 – partial circle of fifths.

The reason behind this disposition can be better explained if we look into the morphology of the diatonic scale. The *interval vector* is a series of six digits which present the intervallic content of any set of pitch classes.[15] The first of the series' digits discloses how many tones at a distance of minor seconds are there in the set; the second digit discloses the major seconds, and so on until the sixth digit, which discloses the tritones. No other digits are required to disclose bigger intervals than the tritone, because, when the subject matter is a *pitch class* set, all those bigger intervals can and shall be reduced to their smaller forms (e.g., any major seventh present in the set is already addressed by the digit of the interval vector that tackles the minor seconds).

The interval vector of the diatonic scale is {2,5,4,3,6,1}, which means that the intervallic relations between all of its tones consist of 2 minor seconds, 5 major seconds, 4 minor thirds, 3 major thirds, 6 perfect fourths and 1 tritone. One might notice that each interval repeats a different number of times, which is a pretty unique feature in the realm of the scales, being shared only by one other heptatonic set (more on that later).

The interval vector also indicates the number of common tones a transposed set will have with the original one – e.g., the first digit of the vector, the one that discloses the number of times a minor second appears in the set, will also represent the number of common tones that a transposition of a minor second will have with the original set.

So, transposing a diatonic scale (whose vector, as we know already, is {2,5,4,3,6,1}) a minor second either up or downwards will translate into a transposed scale with *two* common tones with the original; transposing a major second will mean *five* common tones; transposing a minor third we get *four* common tones, and so on.[16] Most importantly, if we transpose the diatonic scale a perfect fourth we'll get a scale with *six* common tones, that is, all but one. That is why the seven diatonic modes, when disposed by *proximity*, naturally organize in fifths.

Put another way, only one alteration per mode is needed to go from the Lydian mode, in one extreme, all the way until the Locrian mode, in the other (see *Figure 1.6*). Furthermore, that alteration consists of a tone going down by a semitone every single time!

And, as if that was not enough, we can close the system – bridge the gap between the Lydian and the Locrian modes – if we grant it a *mutable root*.[17] That is, regardless of the fact that the Locrian mode is nothing short than *six* alterations away from the Lydian mode, by lowering once again a tone by a semitone – this time, the root of the Locrian mode –we'll get a new Lydian mode (although a semitone lower than the initial one), making it possible to go on with this cycle ad infinitum.

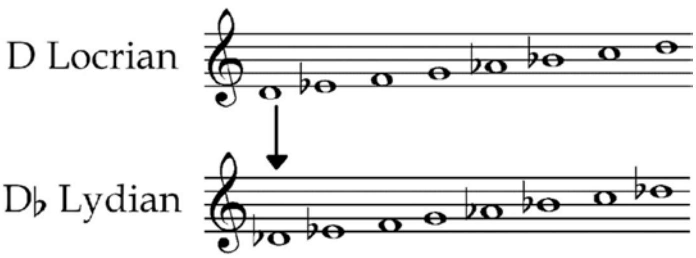


Figure 1.7

This feature is truly unique in the world of musical scales, being shared only by its complement, the pentatonic scale![18] Had I known about that property at the time of my first inquiries, I probably would have declared "case closed" and moved on to something else.

But I did not.

Instead, blissfully unaware that the only thing separating the Locrian and Lydian modes was one alteration (if we grant a mutable root), I went in a completely different direction, which eventually took me to the three families of modes – H1, H2 and H3 – that are, as I mentioned in the introduction, the focus of this thesis. So, for the purposes of this study, mutable roots will not be taken into consideration, and it will be presupposed that the Locrian mode is six alterations away from the Lydian mode. With that in mind, we arrive to the first research question of this thesis.

Chapter 2
Two propositions for a new circle of modes

First research question

In a projection of the seven diatonic modes on the circle of fifths, which five modes best fill the remaining gap between the Locrian and the Lydian ones?

Right from the start I should make clear that the traditional circle of fifths, with its iconic representation of the twelve different key signatures, is *not* the answer to this question. That's because the traditional circle of fifths is the result of the transposition of the *same diatonic mode*[19] twelve times, while the circle of fifths I'm looking for is a representation of *twelve different modes*, having the seven diatonic modes as a starting point.[20]

To know where in the circle of fifths we should project the seven diatonic modes, we must attend to their *prime* transpositions. By prime transposition I mean generally the most straightforward and easily readable transposition of any mode.

It certainly is a subjective matter, but in the case of the diatonic modes, their prime transpositions are without a doubt the ones which display white keys only – the Ionian mode's prime transposition is the one which starts in C, the Dorian mode's is the one that starts in D, and so on – emulating their historical nomenclatures.[21] Therefore, we can logically accommodate each one of the diatonic modes in the circle of fifths by placing them accordingly to the root tone of their prime transposition.

But the remaining gap between the Locrian and the Lydian modes, consisting of the five altered notes of the 12-tone equal temperament (the black keys of a keyboard), will have to be filled by new modes, belonging to families distinct from the diatonic one.[22]

The same way we historically associate a diatonic mode to a specific tone – to the root tone of its prime transposition – we will try to find a mode we can logically associate to each one of the five black keys, and which is not just a transposition of one of the diatonic, "white keys" modes.

Now the research question presented in the beginning arises once again, this time in a more objective, bipartite way:

In a projection of the diatonic modes onto the seven white keys of the circle of fifths according to their prime transpositions, which are the five modes that:

1. *can as logically be projected onto the five black keys?*
2. *can bridge the Locrian and Lydian modes through the black keys just as smoothly as the diatonic modes do through the white keys?*

We can most effectively answer this thorny question by observing two early, embryonic versions of the new circle of fifths we want to build.

Figure 2.1 presents a version of that circle which, once in its complete form, will display all the twelve modes in its *prime* transposition – that will be the *prime index* of the circle. *Figure 2.1* will help us answering the first part of the research question.

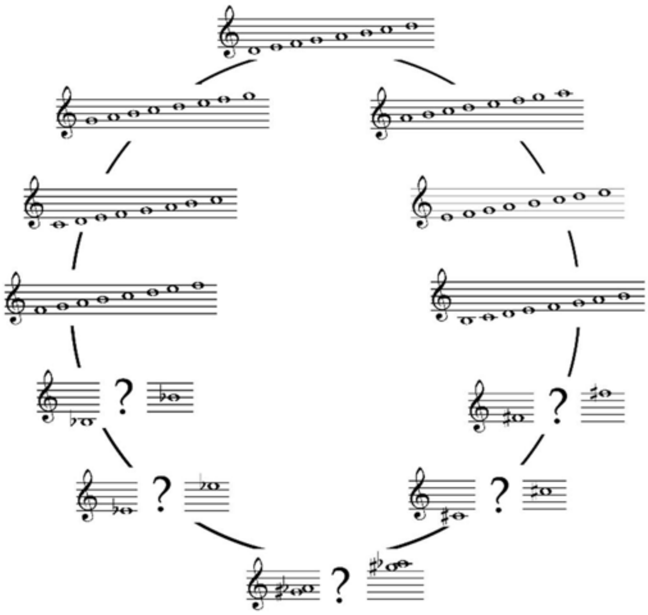


Figure 2.1

Figure 2.2 presents another version of the same circle – this one, once in its complete form, will display the same twelve modes, but transposed to one root tone only, that of their *axis* of reflection – let's call it the *axial index* of the circle. *Figure 2.2* will help us answering the second part of the research question.

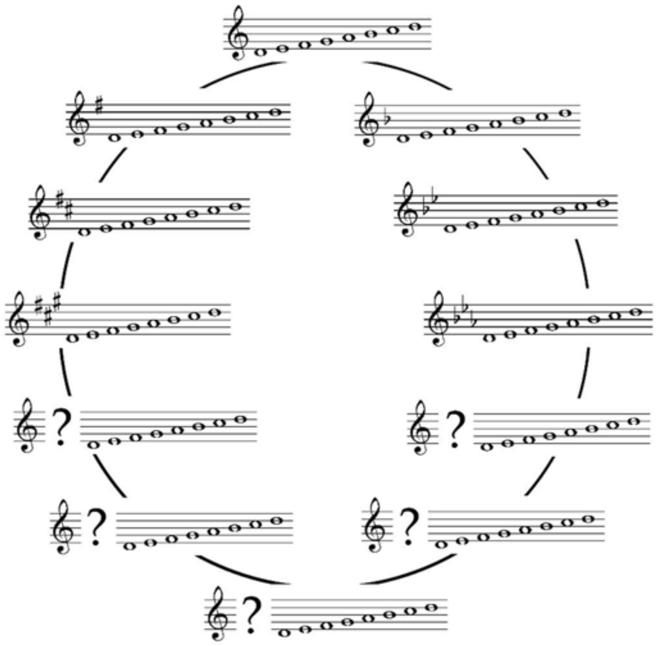


Figure 2.2

As a side note, John Vincent, in his *The Diatonic Modes in Modern Music*, goes through a similar process of organizing the diatonic modes in two lists,[23] one displaying their prime transpositions and the other their axial transpositions (in the latter, he also selects D as the system's axis of reflection). He names the first list "Ordinal Index" and the second "Lateral Index". Those lists consist of a reduced version of this study's prime and axial indexes, respectively, and they naturally inspired the names of my lists.

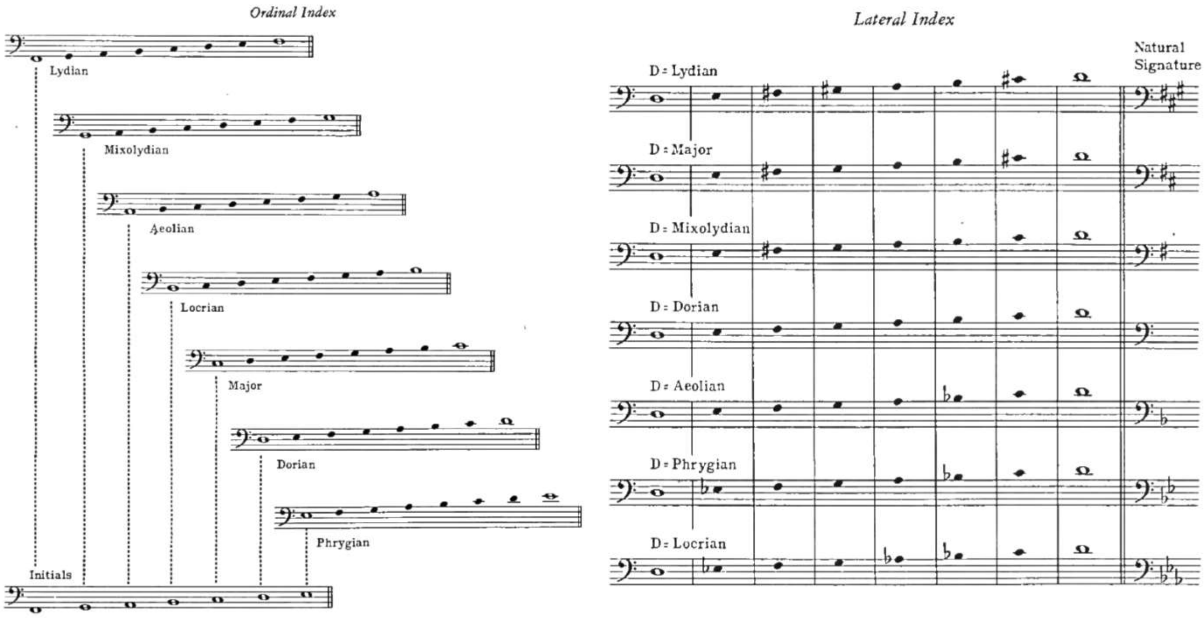


Figure 2.3 – Ordinal and Lateral Indexes of the diatonic modes. Taken from Vincent's The Diatonic Modes in Modern Music.

Now, as we will realize soon enough, not only are the solutions proposed by both indexes the product of an equally valid and logical process, but, most interestingly, they happen to match each other almost perfectly!

A circle's prime index

It was previously stated that it is the circle's prime index which has the answer to the first part of the research question, restated here:

> *"In a projection of the diatonic modes onto the seven white keys of the circle of fifths according to their prime transpositions, which are the five modes that can be projected as logically onto the five black keys?"*

But why is that?

Again, the circle's prime index, once in its complete form, will display all twelve modes in their prime transpositions, and it is thanks to the *prime* transposition of a diatonic mode – let's say, the Phrygian one – that we instinctively know where to project it in the circle of fifths – in this case, the pitch class E. Consequently, if our goal is to project five modes onto the black keys of the circle of fifths, we better focus on their prime transpositions, hence, on the circle's prime index.

At this point, I should dispel the notion that some eventually might have formed that the prime transposition of a mode is the one featuring most white keys. That is indeed the case for the diatonic modes, but not for many others. In some cases, a prime transposition might mean the actual opposite – e.g. the pentatonic modes are better represented by the black keys.

Now that we hopefully warmed up to the idea of a mode having its prime transposition starting on a black key, we can concentrate on the circle's prime index as such. *Figure 2.4* presents the root-tone of each one of our aspirant twelve modes doubled by the octave. In the case of the Ab/G# mode, both tones must be represented as, contrary to all other altered notes, none of these enharmonics have primacy over the other (meaning, both G# and A*b* are at the exactly same distance in the circle of fifths to its axis, D).[24]

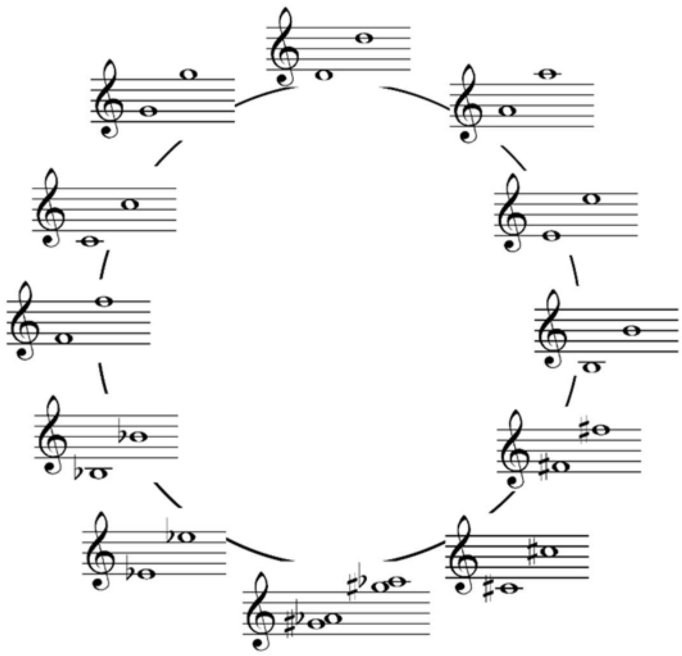


Figure 2.4

The next step in order to find out the five black keys' modes is to fill the gap between the root tones of each mode. Obviously, the diatonic modes, the ones which root falls on a white key, will have to be filled with more white keys. Because of that precedent, we might just as well fill with white keys the gap of our five black keys' modes.

This first attempt to build a circle of modes, displayed in *Figure 2.5*, proves to be a failure – only three new modes were added to the circle, the F# and B*b* ones being nothing more than a transposition of the B and F modes, respectively.

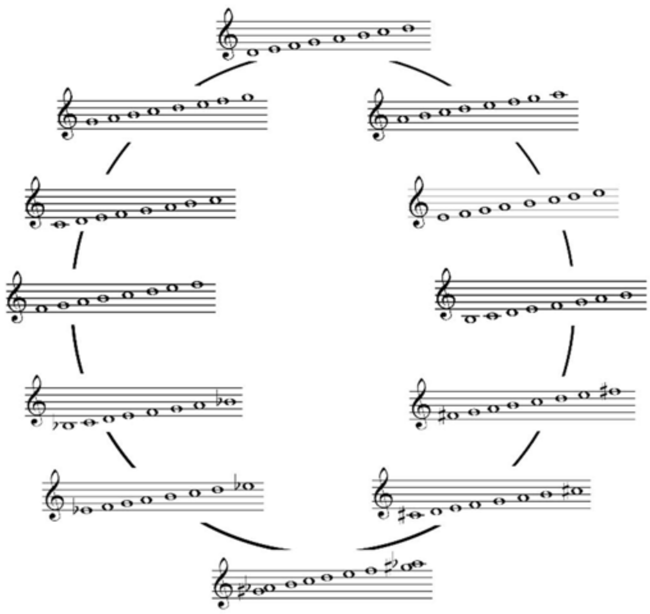


Figure 2.5

Therefore, one must conclude that one single black key is not enough for the F# and B*b*'s prime transpositions to become full-rights modes in this new circle of fifths. An additional alteration is in order.

To know which step we must alter, we have to look once again to the morphology of the diatonic family. There is one feature, not mentioned yet, which brings a diatonic mode even closer to its inverse, that being: whether the mode is shown in its prime transposition or in its axial transposition (D), it will always include its inverse's root tone. E.g., whether an Aeolian mode is in A (prime transposition) or in D (axial transposition), it will always have its inverse's root tone – G – and vice versa.

Figure 2.6

If that feature applies to all diatonic modes, for the sake of consistency, it will have to apply to the new five modes as well. *Figure 2.7* presents the second attempt to build a circle of modes in their prime transposition – this time, I made sure to include in each mode the root tone of its inverse (here represented by a black notehead).

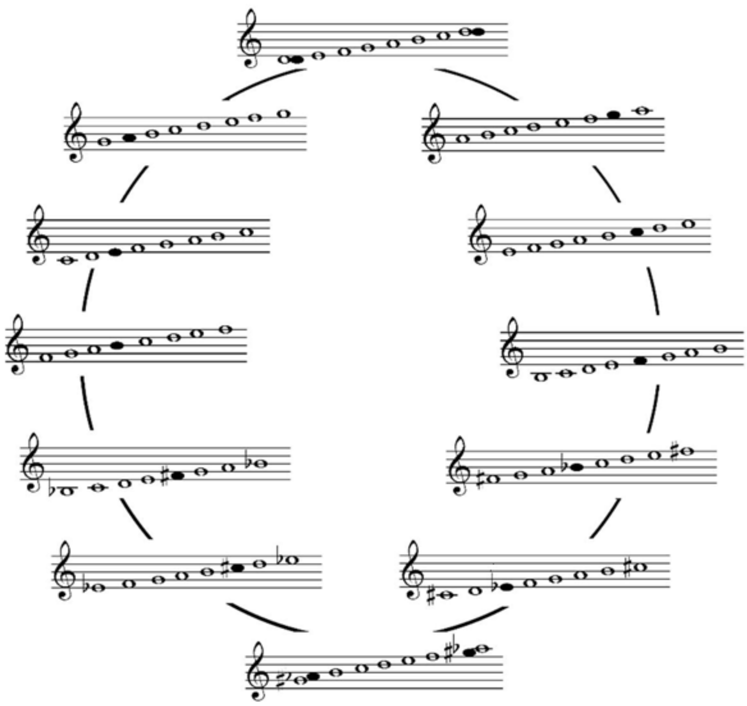


Figure 2.7 – the prime index.

This attempt proves to be much more successful – while nothing changed with the diatonic modes, five new modes were added to the circle. Furthermore, they seem to fit particularly well in the system:

- Just like the diatonic modes, the F# and C# modes are paired with their inverses: the B*b* and E*b* modes, respectively.
- The A*b*/G# mode is its own inverse, as is the case of the Dorian mode on the opposite side of the circle.

Much more could be said about how this solution seems like the right answer to the *first* part of the research question. But first, let us focus instead on answering the *second* part of the research question.

A circle's axial index

First let's reiterate the second part of the research question we're trying to answer.

> *"In a projection of the diatonic modes onto the seven white keys of the circle of fifths according to their prime transpositions, which are the five modes that can bridge the Locrian and Lydian modes through the black keys just as smoothly as the diatonic modes do through the white keys?"*

As noted before, it is the circle's axial index which most promptly provides an answer. The reason being that, for the purpose of evaluating the proximity between a plurality of modes, it is easier to assess their intervallic differences if they all are transposed to the same root tone – this way, one needs only to spot the differences in their key signatures!

Figure 2.8 presents the key signatures of the seven diatonic modes projected by fifths onto the circle of fifths.

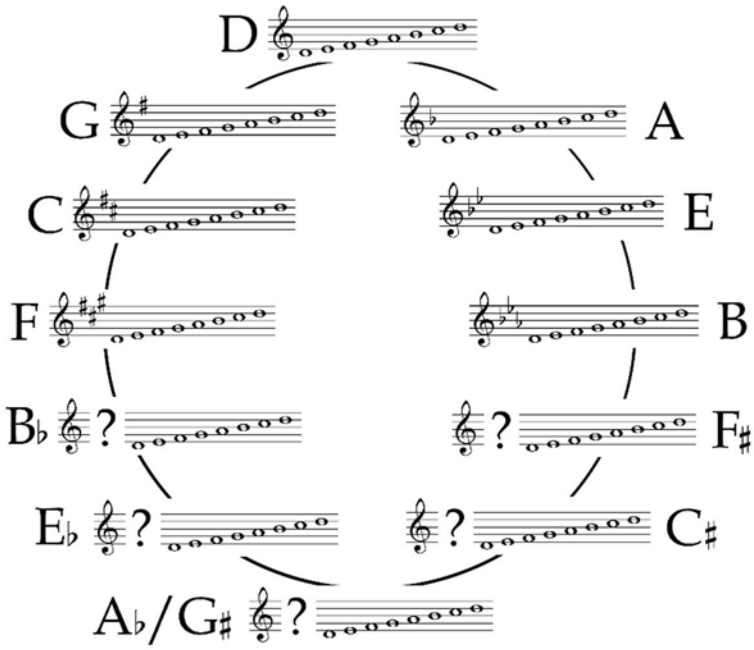


Figure 2.8

At this point, one might stop the logical thread for just a moment and acknowledge the peculiar disposition of the key signatures in the circle of fifths. In fact, as a side note, one might wonder why is it that, in this circle, contrary to the traditional circle of fifths, the sharp-based keys are displayed on the left, and the flat-based keys on the right.

The reason is simple: both circles display their tones *clockwise* by ascending fifths, but while the traditional circle represents the same mode in each one of its 12 transpositions, this new circle will display 12 different modes in the same transposition (their axis's – D).

In other words, while the modes displayed in the traditional circle *share all the same interval pattern but not the transposition*, the modes displayed in this new circle's axial index are the exact opposite – they *all share the same transposition but not the interval pattern.*

As a result, while in the traditional circle the mode without key signature is the one based in C – the Major mode – in the new circle that mode is the one rooted in D – the Dorian one. And while in the traditional circle the next mode in the circle of fifths is G Major – whose key signature is one *sharp* – in this new circle the next mode is D Aeolian – whose key signature is one *flat*.

Curiously, in his already mentioned “Lateral Index”, John Vincent too goes against the norm and organizes the seven modes from the “sharpest” one, the Lydian, to the “flattest” one, the Locrian (see *Figure 2.3*).

Now that that question is out of the way, let us move on to the actual answer to the second part of the research question, starting with offering a hypothesis for the key signature of the F# mode.

The process one can observe for the establishment of a key signature starting with the D mode and ending with the B mode consists of the adding of a flat at a distance of a descending fifth (D – 0 flats, A – 1 flat, E – 2 flats, B -3 flats).

Therefore, it would seem obvious that to arrive to a F# mode one would only need to add to the 3 flats making up the B mode’s key signature another flat at a distance of a descending fifth. However, that flat would mean a lowered root tone (D*b*) and we have already established that no mutable roots are permitted in this circle. Furthermore, the resulting mode would merely be a transposition of the B mode, and that is exactly what we do not want, a transposition of a mode already represented in the circle.

My next hypothesis is to jump in the sequence of descending fifths over the D*b* altogether and add to the 3 accidentals of the B mode the next one in line, G*b*. The resulting mode has a key signature that distinguishes it from any diatonic mode, from which one can only conclude that it belongs to another modal family, and that is precisely what we are looking for.

Meaning that not only do we finally have an acceptable solution for the F# mode, but also a new precedent to take in consideration when filling the remaining vacant spots in this circle. Namely that , in order to create a new mode, one shall simply add to its neighbour’s key signature an accidental at a distance of 2 descending fifths in the case of flat keys or, inversely, 2 ascending fifths in the case of sharp keys.

Bearing that in mind, the C# mode materializes by adding to the F# mode a F*b* (skipping the C*b*).

Figure 2.9

As for the G# mode, one has to enter the double accidentals (F*b* being the last individual flat) – that is, in order to find its key signature, one has to skip over the B*bb* and add E*bb* to C# mode’s key signature.

Now, if we apply exactly the same process to the left side of the circle, the one with the sharp-based keys, we get the following row of key signatures.

Figure 2.10

Most interestingly, the key signature of the G# mode, with its 4 flats and 1 double flat, even if it could not seem further apart from the 4 sharps and 1 double sharp of the A*b* mode's, actually consists of exactly the same tone row, in its enharmonic version. This means that indeed we can place this row of modes in a circle of fifths.

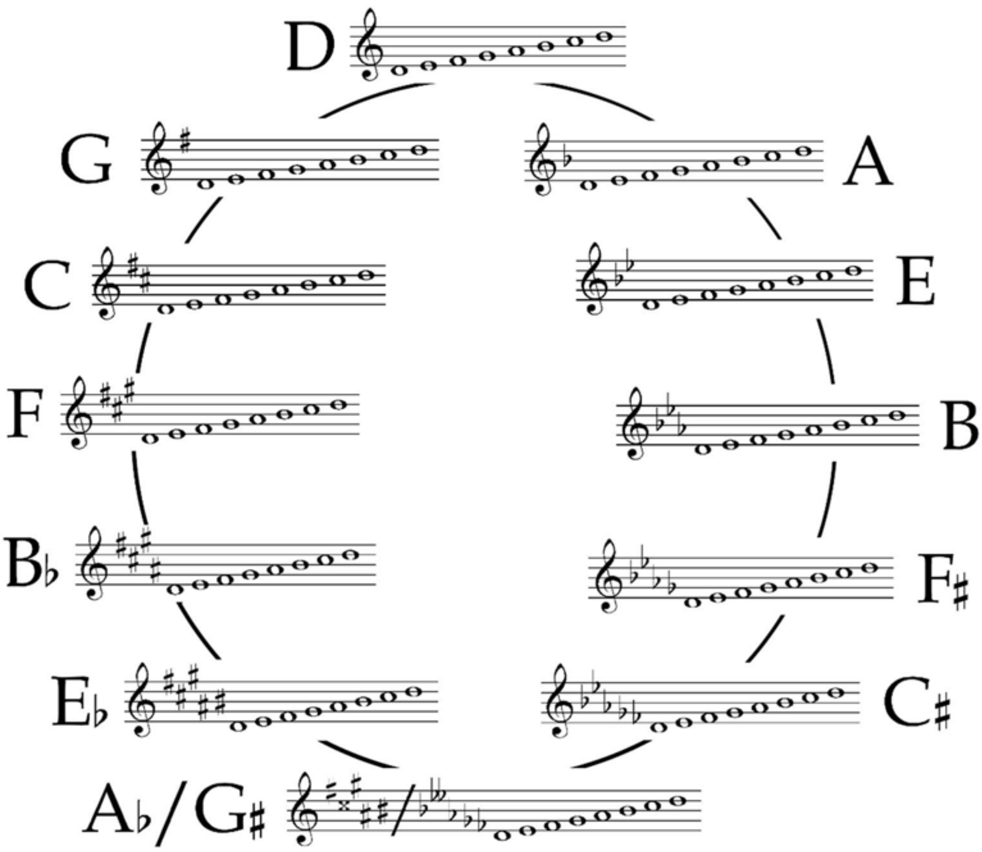


Figure 2.11 – the axial index.

Again, just like the circle built to answer the first part of the research question (the prime index, see *Figure 2.7*), this one provides five new modes to the system, each one of them fitting particularly well in it:

- Once again, the F# and C# modes are paired with their inverses – the B*b* and E*b* modes, respectively.
- And once more, the A*b*/G# mode is its own inverse, as is the case of the Dorian mode on the opposite side of the circle.

But what about the second part of the research question? Do these five new modes bridge the Locrian and Lydian (B and F) modes through the black keys of the circle of fifths just as smoothly as the diatonic modes do through the white keys?

Uncannily, the answer is yes. That is, just like the traditional circle of fifths, it is indeed possible in this circle to travel from any mode to any of its neighbours going through only one alteration at a time.

Now that we completed both indexes, what naturally follows is to compare the results of both, and evaluate how much do they differ with each other, and which one might prove to be the best suited to answer the research question in its entirety, not only one of its parts.

Once again, given the logical process undertook to come up with these two solutions, it shouldn't surprise anyone to verify that the results of both indexes match almost completely – that is to say, their solutions for the F#, C#, B*b* and E*b* modes are exactly the same! The only thing that does not match is their propositions for the A*b*/G# mode. So, which of the solutions proposed by the indexes are we to select to fill that spot in this new circle of modes?

Depending on the answer, one can go into two completely different paths – the consequences of choosing each to be the subject of chapter 4.

But before that, chapter 3 will start from the premise of questioning the necessity for the existence of an A*b*/G# mode in the first place. The reasons for that are three-fold:

1. All but one of the modes presented in this system are heptatonic. The A*b*/G# mode, in both indexes, consists of a hexatonic mode, putting it, one can argue, at odds with the overall logic of the system.
2. Its neighbours (the E*b* and C# modes) do not even need an intermediary – despite what their key signature might tell at first glance, they are already at a distance of one alteration from each other! That is, they share all tones but one – the C# of the E*b* mode (its 7^{th} degree), which transforms into the E*b* of the C# mode (its 2^{nd} degree).[25]

Figure 2.12

3. All the 11 heptatonic modes of the system belong and can be reduced to three families of modes, whose link at the structural level is so strong that leaves no space to an intruder such as the A*b*/G# mode.

It is because of all these reasons that in the next chapter we'll leave the A*b*/G# spot vacant, and divert our attention to the two families of scales we have uncovered while busy in the process of completing both indexes of this new circle of modes. These two families – H2 and H3 – together with the diatonic one – H1 – have a historical bond that goes far beyond the two indexes, and they are the three families of heptatonic scales this thesis is about.

Chapter 3
The three Heptatoniae

Second research question

Before discarding the A*b*/G# mode altogether, let us muse first on the simplicity of design of the new circle of modes' axial index. How alluring is to realize the way it mirrors the traditional circle of fifths – the way one can go from D to A*b*/G# by adding each time one flat to the previous key signature, only to replace all flats by sharps and then go back to D, the starting point, by subtracting sharps. And I chose the verb "mirror" deliberately, as the circumnavigation I just mentioned goes clockwise in the new circle (where flats are represented on the right), while in the case of the traditional circle (where flats are disposed on the left) it goes anticlockwise.[26]

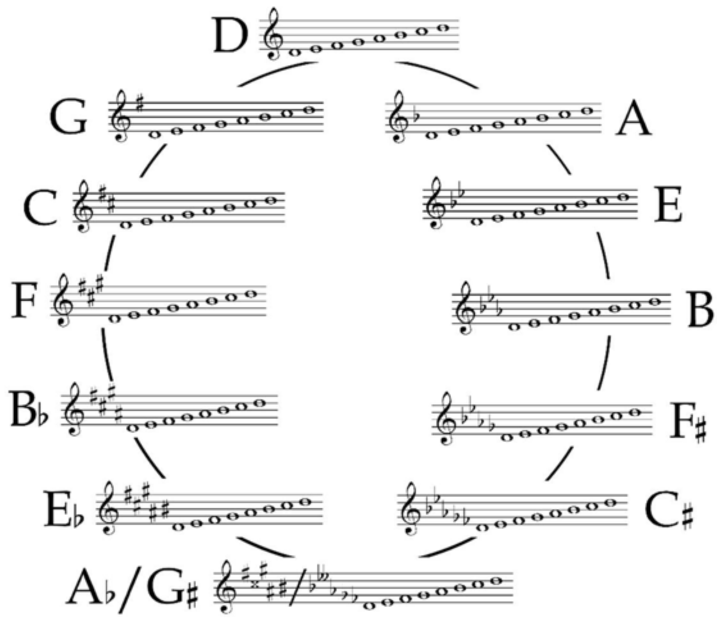


Figure 3.1

The big difference between these two circles of course being the fact that, while the traditional circle of fifths never escapes the same mode (or at best, a pair of them),[27] this new circle of fifths goes from a diatonic mode (the Dorian one) all the way to the whole-tone scale (which carries a completely different historical background, as well as quite distinct harmonic implications) and then back to Dorian, while in the process unveiling two new families of modes!

One thing, though, bothers me with this circle – namely, the fact that, while it features *three* different families of modes (four, if we were still counting with the A*b*/G# mode), only *one* of those families is represented by all its modes, all seven of them. Of course, that is the case of H1, the diatonic family, while H2 and H3 are incomplete, reduced to two modes only. H2 is represented by the F# and the B*b* modes, and H3 is represented by the C# and E*b* ones (needless to say, each one of these two pairs of modes consists of one mode and its inverse). But what about the five modes each of these families contains and which are not represented yet? This circle of modes ought to englobe them as well if it is to reach its true completion. Hence, we arrive to the second research question of this thesis.

How to model a coherent modal system out of the axial index which represents the three heptatonic collections in a comprehensive and proportionate way?

Modes' classification

The prime transpositions of this system's modes can be already observed in the prime index (see page 13) but are represented clearer as *collections* in *Figure 3.2*.

Johnson defines a collection as a "particular arrangement of notes, regardless of tonic, or starting note."[28] Therefore, it is in many ways a synonym for "pitch set" as defined by Rahn.[29] Although the latter is more often used in pitch-class set theory, I will stick to "collection", as "pitch set" has the handicap of carrying a lot of other concepts with it, such as "prime form", which could be confused with "prime transposition".

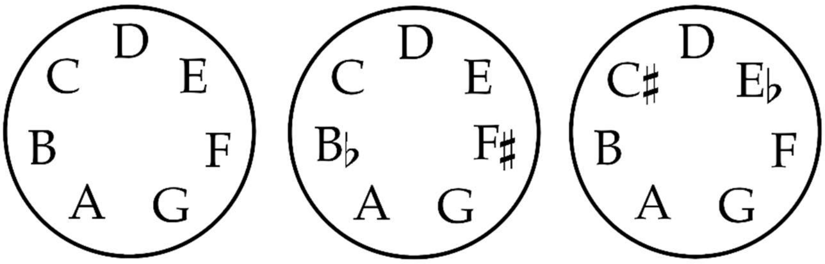


Figure 3.2 – from left to right – prime transpositions of H1, H2 and H3 collections.

In *Figure 3.2.*, the collections are displayed in circular disposition precisely to emphasize the non-disclosure of a root tone. On a side note, while D is disposed on top, that does not make it in any way a root tone – it only makes clear that, as was the case of the diatonic scale, we are once again dealing with collections displaying axial symmetry, and whose prime transpositions' axis falls invariably on D.[30]

Henceforth, the classification of any mode in this system shall proceed in the following manner:

- firstly, a reference to the note which identifies the mode. This note is the root tone of the mode's prime transposition – e.g., "D" will refer to any mode whose prime transposition starts on D (in the case of the diatonic family, it will mean the Dorian mode).
- secondly, a reference to the number identifying the mode's collection (1, 2 or 3).
- third, a reference to the mode's transposition.

Next, a couple of examples are given to make the classification clear.

F1 in C – indicates a mode whose prime transposition starts in F, which belongs to H1 and which is transposed to C – that is – "F1 in C" is C Lydian in other words.

F#2 in F – refers to the only mode in this system whose prime transposition starts in F#, that of H2, and which is transposed to F.

Note that, for example, there is no B2 mode whatsoever, as the prime transposition of the H2 has no B in it, only B*b*. Likewise, there is no B*b*1 mode – that would somehow refer to a non-existent diatonic mode whose prime transposition falls between the Aeolian and Locrian (A and B) modes.

Figure 3.3 lists the 21 modes one can extract from these 3 families – 7 from each, as they are heptatonic – and presents them starting all on the same root tone – D (hence, none of the modes have specified "in D"). Below each mode, it is displayed its series of tones and semitones.

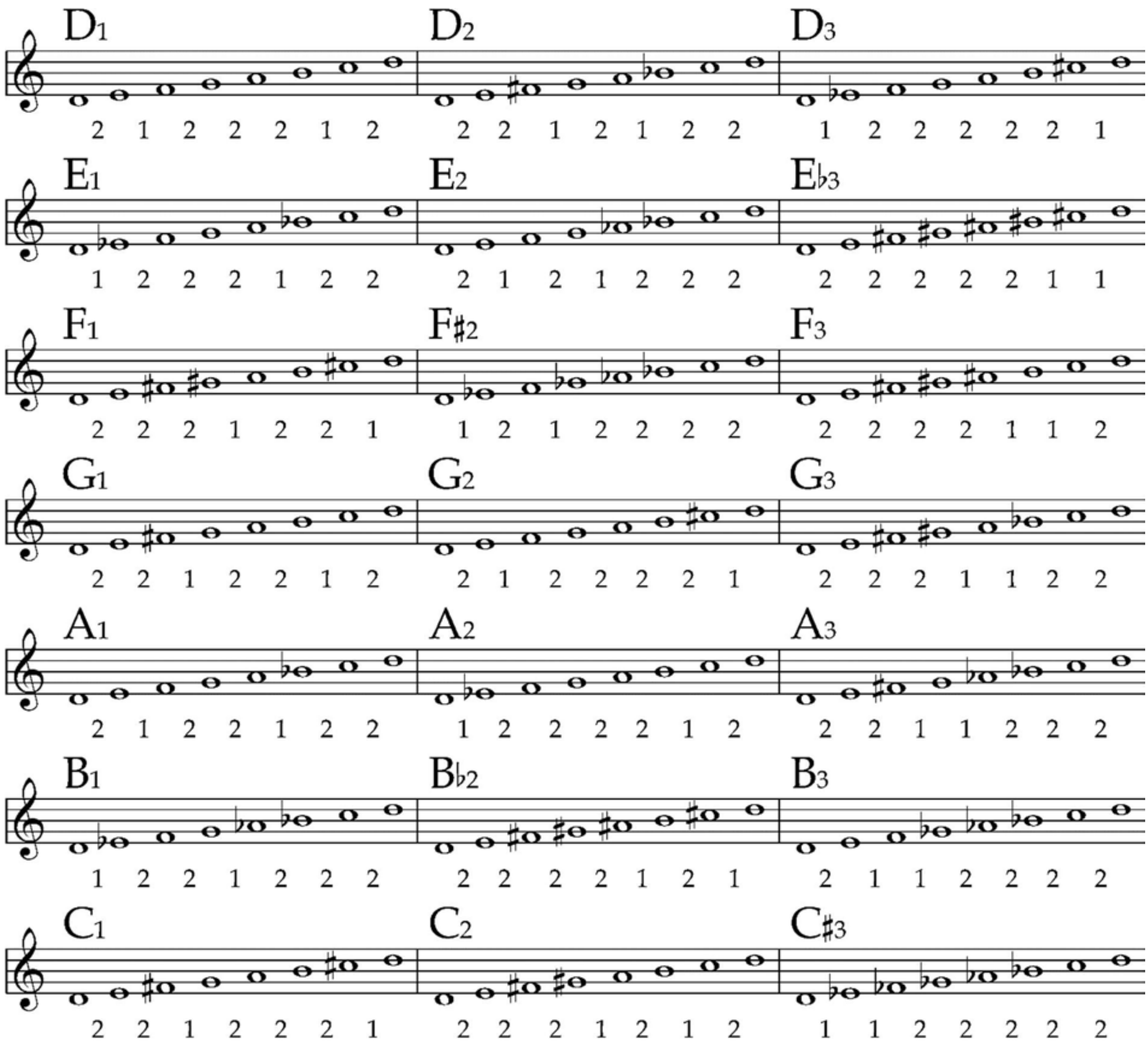


Figure 3.3 – the 1st column displays all 7 modes of H1 (the diatonic collection), the 2nd column displays H2's, and the 3rd column displays H3's. All modes are transposed to D, the axis of reflection of their prime transpositions.

It is at this point that one can formalize a series of facts about these modes. They are all the result of the stacking of five tones and two semitones. What differs is the number of steps separating the two semitones in each collection.

H1 has the semitones maximally separated from each other – depending on our starting point, one must go through two or three steps to reach another semitone. H2 has its semitones slightly rearranged – one or four steps apart from each other. H3 has its semitones once again just retouched – zero or five steps apart from each other.

Now a question inevitably arises – are there any other possible combinations of five steps and two half steps?

The answer is no.

The 21 series aforementioned completely exhaust the possible combinations one can make out of five "2s" and two "1s", meaning that we just stumbled our way into a system that, beyond providing a new circle of fifths that gaps the bridge between the

Locrian and Lydian modes, also treats us with *every single heptatonic mode one can get out of major and minor seconds!*

In how many ways can one arrange two semitones (s) on the seven degrees (d) of a scale? The answer is denoted by C_s^d in the following mathematical expression involving combinations:

$$C_s^d = \frac{d!}{(d-s)!\,s!} = \frac{7!}{(7-2)!\,2!} = 21,$$

where $d! = d \times (d-1) \times (d-2) \times \ldots \times 2 \times 1$. [31]

Of course, I was not the first one to arrive to the conclusion that there are exactly 21 ways of organizing tones and semitones in a heptatonic scale, and I dedicate the final part of this chapter to acknowledge all those whose work I came across that, though working with different frameworks and goals, meandered in the same field as I did and reached the same conclusion.

Extending the circle of modes

And yet, what none of them seem to have noticed or cared was to follow-up that conclusion with a visual organization of all 21 modes based on the proximity of their key signatures.

One might recall me having mentioned the fact that the new circle of fifths – as is currently represented by any of its indexes – seems in a way incomplete, given that H2 and H3 are represented only partially, i.e., by two of their modes. 10 modes are missing, 5 from each of these families. Now that we have figured out the list of 21 modes, we can proceed to assert where in the new circle of fifths the 10 missing modes shall be placed.

But first, we shall establish different degrees of *proximity*. Let us say that distance of 1 is the property of two modes that share the same key signature but for one accident; distance of 2 is the property of having two different accidents in the key signature, and so forth. To make the concept of proximity as restrict as possible, enharmonics will count as differences in the key signature. Both the traditional circle of fifths and the new one have their scales organized in such a way that each of them has a distance of 1 in relation to their neighbours. Our goal is to assess if such a phenomenon is replicable in this extension of the new circle of fifths.

Let us start with D2 in D. To clarify, it is transposed in D because it is the axial index (with all its modes transposed to D, the axis of reflection of their prime transpositions) the one we're going to extend with these 10 other modes.[32]

D2's key signature consists of a F# and a B*b*. It has a distance of 2 with D1, which has no altered notes. On the other hand, it has the biggest possible proximity – distance of 1 – with both G1 and A1. G1 and A1 are themselves more apart from each other – distance of 2 – than from D2. This places D2 somewhere between G1 and A1, that is, in the exact same cardinal point as D1!

Applying this principle to all other modes, one quickly realizes that the modes that share the same prime transposition's root tone (e.g., D1, D2 and D3), share organically the *same spot/cardinal point in the extended circle of modes* as well. That means the extended circle of modes will have 3 modes on each of the cardinal points belonging to G, D and A; 2 modes on C and E; 1 mode on B*b*, E*b*, C# and G#; and, predictably, 0 modes on A*b*/G#. This is the reason behind the symmetrical and progressive ramification of *one* circle with 0 modes on the bottom of the system (on the A*b* /G# spot) into *three* circles with 3 modes on the top (on the D spot).

Figure 3.4 presents the extended new circle of modes based on D.

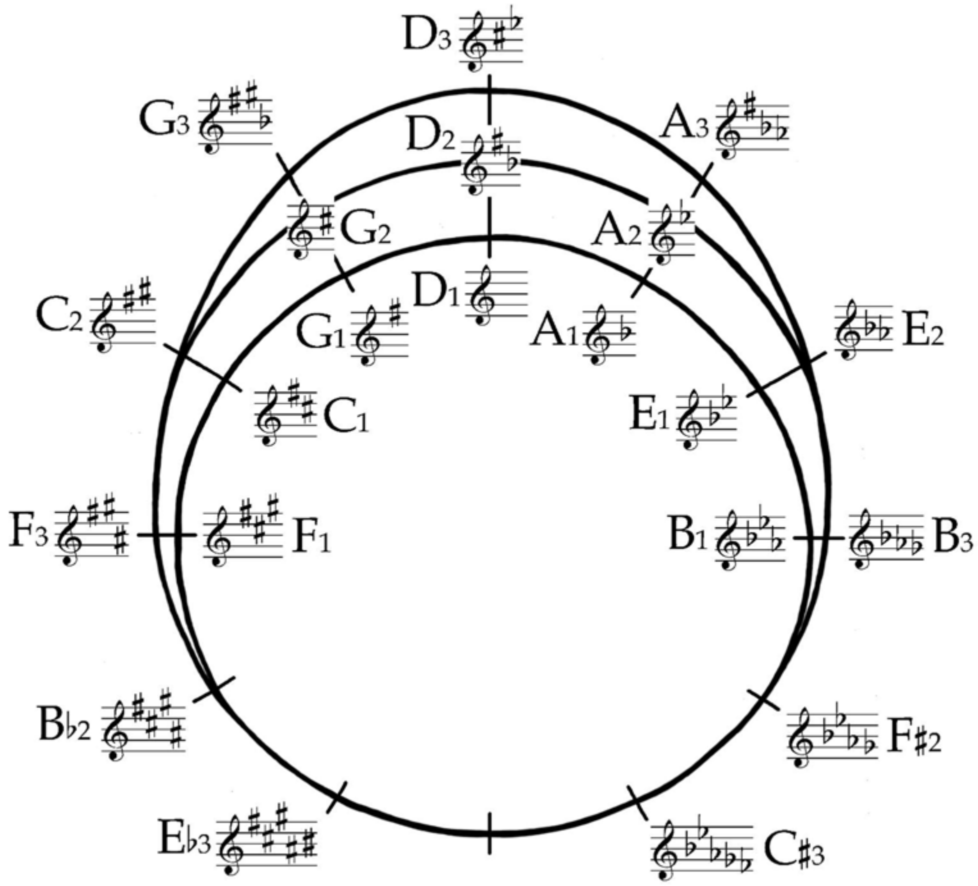


Figure 3.4 – the extended new circle of modes.

Tripling the circle of modes

The extended circle of modes presents the key signatures of all 21 modes one can extract out of H1, H2 and H3. Put another way, it gives away every single heptatonic mode one can get from stacking major and minor seconds over a shared root tone. Therefore, all 21 modes have one note in common – in the case of *Figure 3.4.*, that note is D.

But what if we were to enlarge once again this system by making it encompass, not only one, but all 12 transpositions of these 21 modes? The fact that we are dealing with modes featuring axial symmetry instead of rotational symmetry obviously means that we cannot count with repeated intervallic patterns, therefore, nor can we count with two transpositions of the *same* mode sharing key signatures (as we would in the case of the former).

And yet, it only takes some additional 15 key signatures to go from a system like we have in *Figure 3.4* (featuring 21 modes based only on one root tone) to a system encompassing all 12 transpositions of each one of those 21 modes. Of course, that derives from the fact that the same key signature is shared by *different* modes in *different* transpositions. But why exactly does that translate in a grand total of 36 key signatures?

The explanation is simple – that is the result of the multiplication of the 3 collections by their 12 transpositions.

To put it another way, as all tone combinations featuring D have already been sorted out, one needs now to find only those which do *not* include D; that is, they will have to feature both C# and D# (or their enharmonic equivalents). One step is already sorted out; therefore, one needs only to find the number of possible combinations of the other six steps (four steps and two half steps, to be precise).

That is, in how many ways can one arrange two semitones (s) on the remaining six degrees (d) of a scale? To sort that out, we can resort to the same mathematical expression we used on page 21:

$$C_s^d = \frac{d!}{(d-s)!\,s!} = \frac{6!}{(6-2)!\,2!} = 15.$$

Figure 3.5 presents the newest version of this system, the now tripled circle of modes, with its 36 key signatures equally divided by the 12 cardinal points. Because one can extract 7 different modes from each key, nothing less than *252* different modes "harmonically coexist" in this system.

Our system of modes seems now finally complete. And yet, the disclosure of its design, instead of providing a satisfactory closing to this thesis, only raises more questions, by far the most important being:

How far away from the traditional circle of fifths are we really? And is that enough to

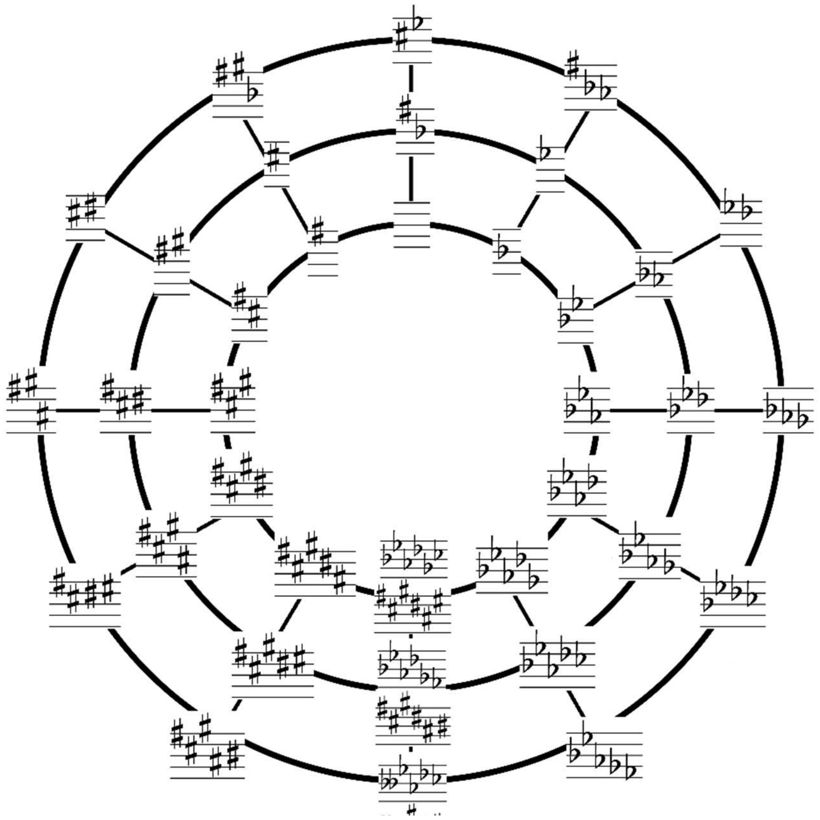


Figure 3.5 – the triple circle of fifths (a treble clef is implicit behind every key signature).

truly justify the mirrored disposition of the key signatures (with the flats on the right and the sharps on the left)?

As one might recall, this is not the first time we pose this question – we did so when dealing with the axial index and the extended circle of modes. This time, however, I will risk a quite different answer.

Until now, we have been dealing with sorting modes out of a common root tone. Once that root tone was appointed (and we always favoured D, being as it is the axis of the diatonic space), the transposition factor became completely out of the equation, and that allowed us to organize the modes on the circle of fifths taking into account the letters which identified, *not* their transposition, but their mode (e.g., F for Lydian). That is, as we know, the reason behind the disposition of the alterations on the new circle, which mirrors the traditional one.

However, now that we have arrived at the triple circle of modes, we are dealing no more with modes only, but with their transpositions as well.

What does that mean? It means this triple circle finds its space in a grey zone somewhere between the traditional circle of fifths (which, in its strictest interpretation, deals with different transpositions of one mode) and the new circle of fifths (which deals with one transposition of different modes).

Now, where on the grey zone one chooses to place this system – whether closer to the traditional circle or closer to the new one – depends on which of the two following interpretations one finds the most convincing.

If one chooses to interpret this circle as the 12 transpositions of the same three modes (each represented by a circle); and if this system is indeed a circle of *fifths;* then it naturally follows that one will favour the version which clockwise organizes its transpositions by fifths and, consequently, its modes by fourths. The result of this interpretation will be a system which replicates the key signatures' disposition of the traditional circle of modes, with the sharp keys on the right and the flat ones on the left. We can call this the

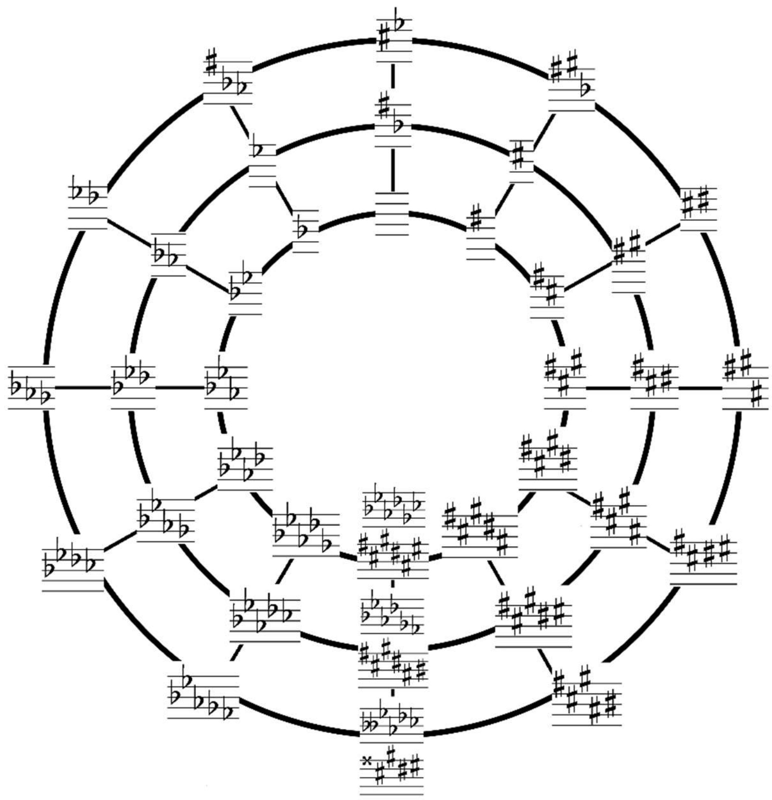


Figure 3.6 – «transposition over mode» interpretation of the triple circle of modes.

transposition over mode interpretation.

Whilst if one chooses to interpret this circle as the sum-up of all 12 transpositions of the extended circle of *modes*; and, once again, if this system is to be a circle of fifths; then it inherently follows that one will favour the solution which clockwise organizes its modes by fifths and, consequently, its transpositions by fourths. The result of this interpretation will be a system which replicates the key signatures' disposition of the extended circle of modes, with the sharp keys on the left and the flat ones on the right. We can call this the *mode over transposition* interpretation (see *Figure 3.5)*.

Both interpretations are however oversimplifications of the true potential of this system. If anyone is to take seriously this triple circle as the basis or starting point for the harmonic constructions of a piece, I would guess that he or she will do so with the intention of expanding *both* vertically (through transposition) and *horizontally* (through modal alteration) those harmonic constructions. At least, that is the way I interpret this system and that is the reason why I deem both designs equally as valid.

Nevertheless, I would still add that the *transposition over mode* version should be the one adopted in case we were determined to attribute a pitch class to each of the 12 cardinal points of the triple circle. The reason why is that this way one highlights not only the root tone of the three palindromes one can extract from each of the cardinal points, but the centre of the "area of influence" of a particular pitch as well – that is, the centre around which all pitches of the same class are symmetrically spread throughout the key signatures. Of course, the same logic could be used to attribute a pitch class to each cardinal point of the *mode over transposition* version as well, but those pitches would be organized by fourths, instead of fifths, and one has to admit that that goes a bit against the whole logic of a circle of fifths in the first place.

For all these reasons, from now on the *transposition over mode* version will be the standard of this study, starting with *Figure 3.7*, which consists of the same triple circle of modes, although this time the lines connecting the key signatures represent proximities of 1; that is, every mode is connected to those which it is closest – to those with which it shares the key signature except for one accident.

From the observation of the third circle of the system – the one concerning the H3 family of modes – one will realize that each one of its modes has 6 notes in common (out of 7) with 5 other H3 modes. E.g., D3 shares all notes but one with E3, F#3, A*b*/G#3, B*b*3 and C3. And yet, they are not connected by the lines representing proximities of 1. That's because the way I defined "distance of 1" discards all those modes as the similarity of their key signatures depends on a series of enharmonic equivalences. Of course, the reason why I defined it that way was merely to avoid overcrowding the triple circle with tangled lines. But that shouldn't blind us from the curious relationship between H3's transpositions, which are starkly divided into two groups, inside which they share all notes (enharmonically speaking) but one, and outside of which they have no notes in common but one.

To sum up, the triple circle of modes (as desplayed by *Figure 3.7*) together with the extended circle of modes (see page 23) make up the answer advanced by this study to the research question that triggered this chapter, and which I echo here:

How to model a coherent modal system out of the axial index which represents the three heptatonic collections in a comprehensive and proportionate way?

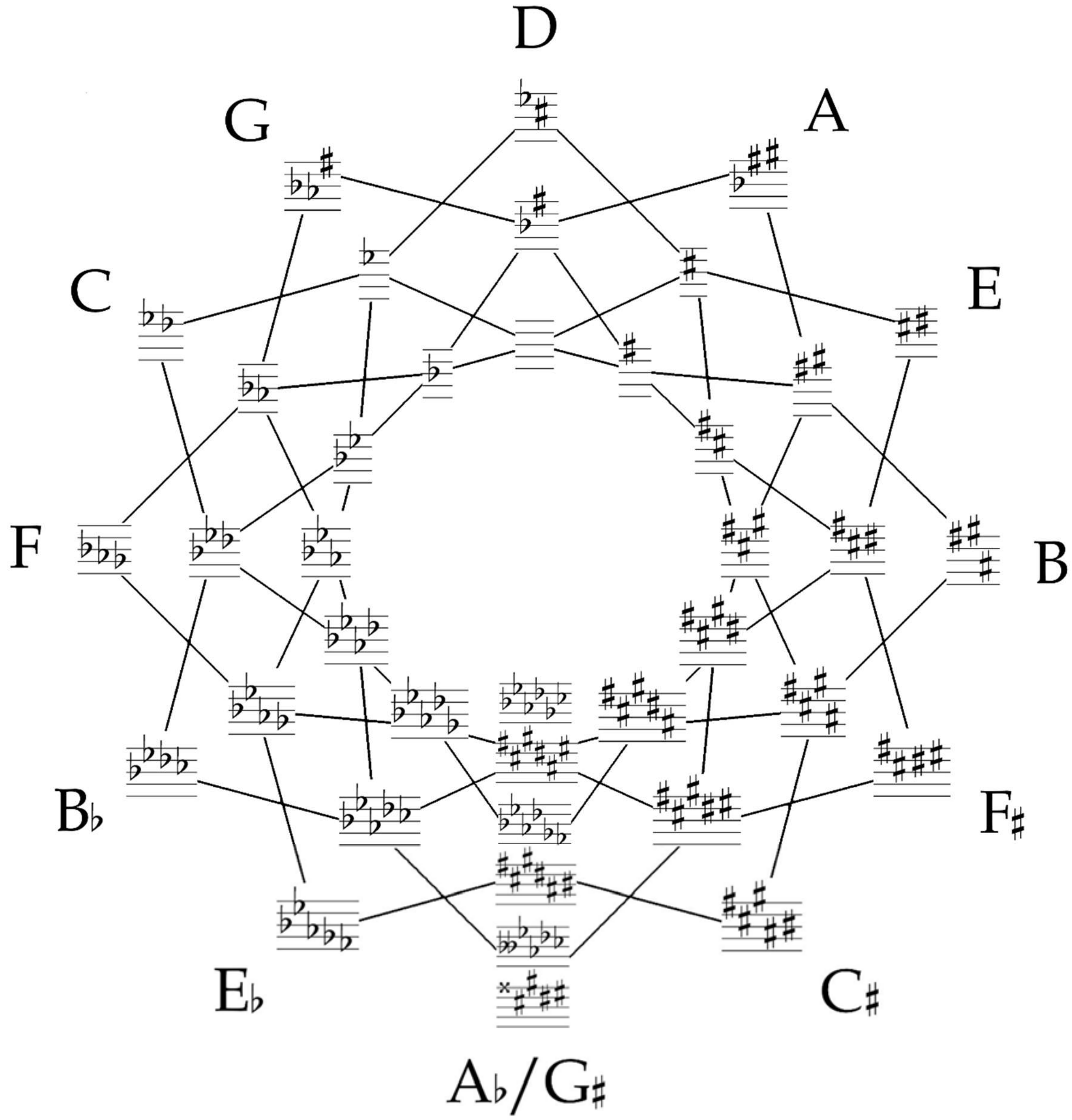


Figure 3.7 – the triple circle of modes, with lines linking those keys that feature distance of 1.

Now, I have mentioned earlier in this chapter that this study is by no means the first one to address neither the existence of the three families of scales represented by the triple circle of fifths, nor the properties that link them together, in a bond that isolates them against all other existing heptatonic scales. After all, these scales, under the name Heptatonia Prima, Secunda and Tertia, go as far as making up a considerable part of the Wikipedia article[33] about heptatonic scales.

Still, I was not able to trace any sign of other people, musician or theorist, who went as far as exploring the relationship between the transpositions and modes of those families by crystalizing them in a circle of fifths, and that's even more surprising given the weight of some of the names which dealt with these modes a long time before I did. Anyhow, next I go through the names whose work, as far as I am aware, came closer.

Heptatonia Prima, Secunda and Tertia

Michael Keith, an American mathematician, took his knowledge of an area of mathematics known as combinatorics to write *From Polychords to Pólya: Adventures in Musical Combinatorics* in which he approaches a variety of issues belonging to musical set theory, most dealing with counting and classifying chords and scales. On the chapter about "scale-counting problems" he highlights 21 modes, which he calls the 21 *diatonic scales*, and divides them into three groups – A, B and C – about which he says the following:

> *"Group A is the most commonly used, and includes the major scale. The scales in group B are somewhat less frequently heard, and group C is the least common of all."*[34]

As one may have already guessed, those A, B and C groups consist of just another name for H1, H2 and H3 – the families of scales this thesis is all about. Note the paralelism between *Figure 3.3*, where I catalogue the 21 modes I extracted from the circle of modes, and *Figure 3.8*, where Keith lists the exact same modes; the only thing that changes being the root tone which they are based on (mine start in D, Keith's start in C).

Intervals	Scale in the key of C	Name
	Group A	
2 2 1 2 2 2 1	C D E F G A B	Major
2 1 2 2 2 1 2	C D Eb F G A Bb	Dorian
1 2 2 2 1 2 2	C Db Eb F G Ab Bb	Phrygian
2 2 2 1 2 2 1	C D E F# G A B	Lydian
2 2 1 2 2 1 2	C D E F G A Bb	Mixolydian
2 1 2 2 1 2 2	C D Eb F G Ab Bb	Natural Minor
1 2 2 1 2 2 2	C Db Eb F Gb Ab Bb	Locrian
	Group B	
1 2 1 2 2 2 2	C Db Eb Fb Gb Ab Bb	Super-Locrian
2 1 2 2 2 2 1	C D Eb F G A B	Melodic Minor
1 2 2 2 2 1 2	C Db Eb F G A Bb	B3
2 2 2 2 1 2 1	C D E F# G# A B	B4
2 2 2 1 2 1 2	C D E F# G A Bb	B5
2 2 1 2 1 2 2	C D E F G Ab Bb	B6
2 1 2 1 2 2 2	C D Eb F Gb Ab Bb	B7
	Group C	
1 1 2 2 2 2 2	C Db Ebb Fb Gb Ab Bb	C1
1 2 2 2 2 2 1	C Db Eb F G A B	C2
2 2 2 2 2 1 1	C D E F# G# A# B	C3
2 2 2 2 1 1 2	C D E F# G# A Bb	C4
2 2 2 1 1 2 2	C D E F# G Ab Bb	C5
2 2 1 1 2 2 2	C D E F Gb Ab Bb	C6
2 1 1 2 2 2 2	C D Eb Fb Gb Ab Bb	C7

Table 4.4. The 21 diatonic scales

Figure 3.8 – Taken from Keith's From Polychords to Pólya: Adventures in Musical Combinatorics, p. 95.

It is noteworthy that Keith generalizes the term "diatonicism" to include all modes one can extract from the three families listed above. That is far from being the consensual application of the term; and yet, this term was never consensually used throughout History. Apparently not even inside the same institution - as a case example, while Oxford Music Online defines a heptatonic scale as diatonic "when its octave span is filled by five tones and two semitones, with the semitones maximally separated",[35] the 2011 Oxford Companion to Music describes diatonicism as using "exclusively notes belonging to one key", adding the "proviso that the alternative submediants and leading notes of harmonic and melodic minor allow up to nine diatonic notes, compared with the seven available in a major scale."[36] The latter's definition matches to some extent Johnson's observation that "other sources and contexts sometimes use the term more loosely to include other seven-note collections as well, such as the harmonic minor and the ascending melodic minor."[37]

If we are to accept that looser definition of diatonicism, then H2 is to be considered diatonic, as one of its modes consists of the ascending melodic minor scale. And if in the process we did give up on the whole idea that "diatonicism" means that the two semitones are maximally separated from each other, then why not consider H3 diatonic as well?

As far as I know, none besides Keith have suggested to do so, and yet it is interesting from a theoretical point of view to entertain that idea for a moment – suddenly, we are able to define diatonicism neither as strictly as Oxford Music Online does, nor as loosely as the same university's Companion to Music does, but as the feature of *any heptatonic scale whose octave span is filled by five tones and two semitones*, period, without any mention to how distanced are the semitones from each other.

Of course, in this redefinition of diatonicism, not only would the 21 scales listed above be diatonic in effect, but they would be *exclusively* so – no other scale could claim to be diatonic because no other scale matches that definition.

Thorvald Otterström, a fin de siècle Danish-born American composer, also arrived to the same three families of scales, although in yet another way. In his fascinating but obscure treaty *A Theory of Modulation*,[38] he goes through the following process:

(a) First, he converts the 7 diatonic modes into rows of numbers;
(b) He then transposes them to the same starting point (in a process eerily similar to the logic behind the prime and axial indexes of this study);
(c) He proceeds to repeat (b) three times.

(a)	(b)	(c)
2212221 Ionian	2212221	221222122122212212221
2122212 Dorian	2122212	212221221222122122212
1222122 Phrygian	1222122	122212212221221222122
2221221 Lydian	2221221	222122122212212221221
2212212 Mixolydian	2212212	22122 1222122122212212
2122122 Aeolian	2122122	21221 2221221222122122
1221222 Locrian	1221222	12212 2212212221221222

Figure 3.9 – taken from Otterström's A Theory of Modulation, pp. 130, 131.

He then reads the (c) table "diagonally from left to right down (or vice versa)"[39] and arrives to a H2 mode. As he explains – "we will get seven new scales in which the factors are identical with those of the original seven scales – five 2's and two 1's; but the arrangement will be different."[40] Otterström concludes the process by using the same mathematical formula I did on page 21 to prove there are 21 scales with those exact same "factors".

Just as Keith, he goes on to list those 21 modes rooted on C, although he does not go as far as honour them with the label "diatonic", instead preferring "permutation of the major scale".[41] Nevertheless, Otterström was a true pioneer, insofar as he was the first one, as far as I know, to report the link between these 21 modes, and he did so more than half a century before Keith!

But by far the most precious contribute to the theory of this harmonic system comes from Lajos Bárdos, one of the biggest names of the Hungarian musical scene of the 20th century, even if not as recognized as his teacher's (Zoltan Kodály), his colleague's (György Ligeti) or his pupil's (György Kurtág) at the Franz Liszt Academy of Music.

In an extensive paper written in 1963 and titled *Heptatonia Secunda, A unique tonal system and its modes in the works of Zoltán Kodály*, Bárdos goes through arguably one of the deepest theoretical explorations of Kodály's harmonic language. He proves, through a never-ending supply of musical examples, the ubiquity in the composer's works of H2-based harmonic constructions (such as progressions, cadences, pendulums), this way establishing H2 as a tonal system in its own right, that could perfectly go toe-to-toe with H1's.

Later on the paper, he also points to another type of scale, "the third possible (without augmented-second) seven-tone system",[42] which he names Heptatonia Tertia. Implied in all these labels is of course the existence of a "Heptatonia Prima" - the diatonic family – even though Bárdos never mentions it as such (the first reference to such a term probably comes from an analyses of a Bartók's piece by László Somfai).[43] It goes without saying that H1, H2 and H3, the terms I use in this study to refer to the three families displayed on the triple circle of modes, are all short versions of the terminology envisaged by Bárdos and Somfai.

Still, the sole focus of Bárdos' paper was to analyse Kodály's music exclusively through the lens of a H2-based "tonal-idiom", which he portrays as being simultaneously new and old. New because only in the beginning of the century, he argues, did composers (such as Kodály and Bartók, but also Debussy) start to explore its mechanics in a serious, even if not necessarily totally conscious[44], fashion. Old because that same tonal language can be traced, at least in part, to folk music, particularly Hungarian one.

Bárdos starts his paper by pointing to a specific set of tones profusely used by Kodály in different compositions, and which differs from the diatonic prime collection in one tone only – C becomes C#. He goes on to claim that the seven-tone system this pitch collection belongs to has a proven practical application, in the sense that all its seven modes are used to some extent by Kodály in many different contexts. Bárdos then expands vertically this yet non-transposed system by considering the "translocated orders" (by which he means "transpositions") of the initial set of tones.

A big chunk of the paper explores the specific handling by Kodály of these modes on the harmonic and contrapuntal level; and hypothesises to what degree Hungarian folk music had an influence on those aesthetic choices. To someone who has an interest in these extensions of diatonicism, Bárdos' paper on Kodálys' music is priceless, as an early and comprehensive statement on the practical possibilities (and by that, I mean the compositional applications) of the H2 collection. Still, Kodály is far from being the only one to take to good use such a system (and a whole study about how other musical genres such as jazz approach this tonal order is much in need).

I could not but scratch the surface of Bárdos' paper here, but now we must go back to the end of chapter 2 (see page 17) where one question was posed – which of the solutions proposed by the aforementioned indexes are we to select to fill the A*b*/G# spot in this new circle of modes? One might recall that the only difference between the prime index and the axial index of the new circle of modes is their suggestions for the A*b*/G# spot – through the prime index we arrive to an esoteric palindromic scale with 2 augmented seconds; while through the axial index we arrive to the well-known whole-tone scale.

Next chapter will be two-fold. First, we will concentrate on the axial index and analyse the connection of H1, H2 and H3 with the whole-tone scale's family, which we from now on might as well label H4 (although this time the "H" stands for "Hexatonia"), and we will do so resorting to a series or properties developed by the field of diatonic set theory. Second, we will briefly turn our focus to the prime index and its suggestion of a different scale for the A*b*/G# spot, we will rethink the process behind that suggestion and stretch it to its very limit.

Chapter 4
A fourth Heptatonia?

A closed system

Consider three bracelets with 12 beads each. How to explain the changes in pattern from bracelet (a) to (c), and which bracelets will come next?

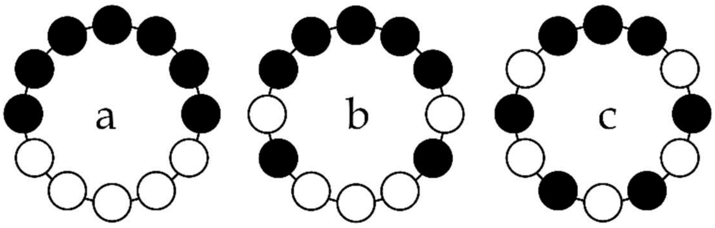


Figure 4.1 – (design taken from Ian Ring with his permission).

One way is to establish a posteriori that each black bead moves downwards whenever the bead below it is white.

But what about the top black bead? To which side shall it move downwards if both its neighbours become white at the same time? We need another rule – the top black bead, having no reason to choose one side over the other, must simply split into two and choose both.

With these two rules, we get four more patterns, after which the process ends.

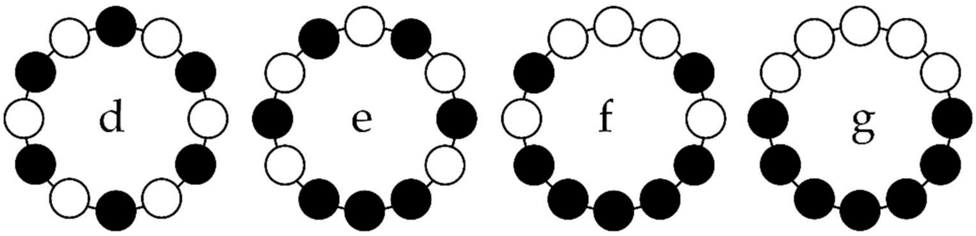


Figure 4.2

It should be noted that this series of patterns is strictly symmetric – the first bracelet, (a), mirrors the last, (g); the second mirrors the second last; the third mirrors the third last; and the fourth bracelet, (d), the middle one, is symmetric to itself, both vertical and horizontally.

Now, if instead of bracelets we think of a circle of fifths[45] with D at the top, and if instead of black beads we think of pitches, we will get the following scales:

(a) D E F G A B C D — D1 in D
(b) D E F# G A Bb C D — D2 in D
(c) D E*b* F G A B C# D — D3 in D
(d) D E F# G# A# C D or G# A# C D E F# G# — D4 in D or in G#
(e) G# A B C# D# E# F*x* G# — D3 in G#
(f) G# A# B# C# D# E F# G# — D2 in G#
(g) G# A# B C# D# E# F# G#, — D1 in G#

Figure 4.3

The point being that the tonal system proposed by the axial index is a closed one, consisting of the three families of modes we know as H1, H2 and H3; and by the two existing transpositions of the whole-tone scale (H4).This hexatonic scale might then be considered the gate that seals the three *Heptatoniae* from the interference of any other modal family; as the operation behind bracelets (d) and (e) shows, there is nothing beyond H4, except more transpositions of the same Heptatoniae.

Figure 4.4 is yet another extension of the new circle of modes (I assure it is the last one), this time englobing all 12 transpositions of H4 as well (maybe it is now time to rebaptize it as "quadruple circle of modes"). Just like *Figure 3.7*, representing the triple circle of modes, this extended system has its keys connected by lines representing "distances of 1".

Of course, we all know that, given its rotational symmetrical nature, there are only 2 distinguishable transpositions of the whole-tone scale. Therefore, the quadruple circle of modes ends up presenting those 2 pitch collections 6 times each. Unless we interpret the accidents of those key signatures in a new, creative way – maybe from a perspective of a 12-tone *unequal* temperament system) we risk being caught in a web of modal redundancy.

Nonetheless, since the establishment of 12-tone equal temperament, composers have been writing, for example, in D*b* and C# Major interchangeably, and they never disregarded one of the scales because its notes were enharmonically the same. Instead, they recognized that one tonality would lead more easily to a particular tonal sphere, while the other would lead to the opposite one. The same can be said of the whole-tone scale (although even I admit that one thing is to have two enharmonic ways of representing a collection, another is to have six…). Still, it is interesting to observe how smoothly H4's collections prolong and fit into the pre-existent system.

From the observation of *Figure 4.4* one reaches another striking conclusion – each pitch collection (represented by a key signature), independently of where it belongs in the quadruple circle of fifths, *has exactly four other collections at a distance of 1 alteration*:

- Each H1 collection displays the closest proximity with the two H1 and two H2 collections which are at a distance of a 5th.
- Each H2 collection equally displays the closest proximity with its neighbours at a distance of a 5th, although this time the latter belong to H1 and H3.

- Each H3 collection displays the closest proximity with its neighbours at a distance of a 5th which belong to H2 and H4.
- Each H4 collection displays the closest proximity with its H3 neighbours at a distance of a 5th and with its H4 neighbours at a distance of two 5ths. Of course, each H4 collection is enharmonically the same as five other H4 collections, but only two of them are at a distance of 1 alteration.

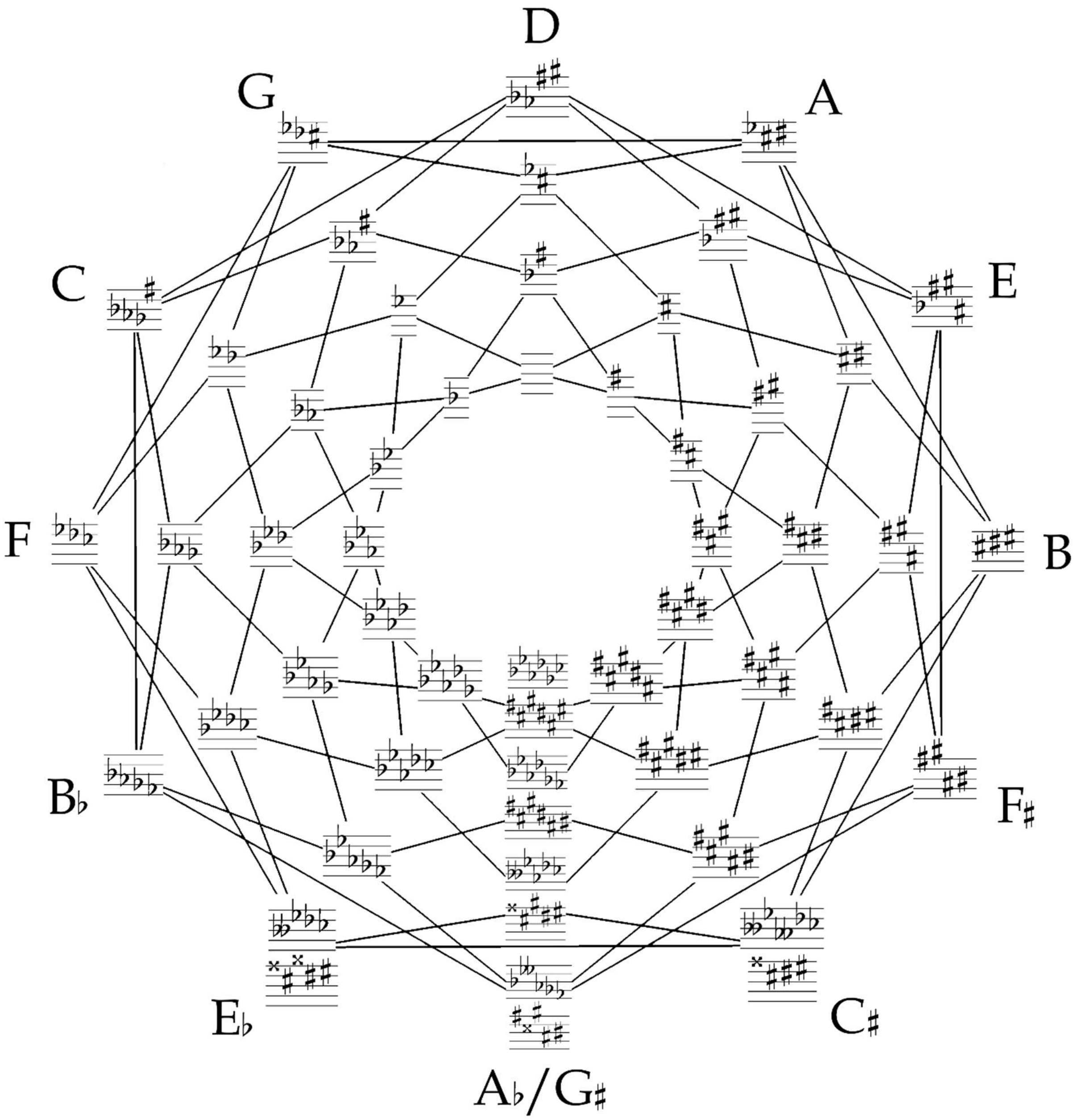


Figure 4.4 – the quadruple circle of modes.

What follows is an overview of some of the most interesting properties on which the entire field of diatonic set theory rests its case. Music theorists and mathematicians alike, sharing a special interest in finding out what made the diatonic scale "special", uncovered many of those properties not so long ago – some as late as 2019.

My goal is not to simply acknowledge the existence of those properties, but to apply them to the four families suggested by the axial index, the ones that eventually culminated in the quadruple circle of modes. Hence, we arrive to the last main research question:

What do the properties attesting the qualities of the diatonic collection tell us about the other three collections making up the quadruple circle of modes?

Deep scales

The deep scale property is fairly straightforward – first explored in 1967 by Carlton Gamer,[46] it is that of any set of tones whose interval vector (see page 6 and 7) consists of unique values. Only very few scales feature this property – Ring identifies only two[47] (those being the diatonic collection and one of its variants with less one note), although a conclusion on this matter is contingent on the definition of "scale". Still, none of our collections besides H1 showcases this property in any way.

And yet, it is interesting to go through their interval vectors and analyse the differences.

H1 {2,5,4,3,6,1}
H2 {2,5,4,4,4,2}
H3 {2,6,2,6,2,3}
H4 {0,6,0,6,0,3}

What can we learn from it?

First, that our circle of modes consists of a two-way bridge between the tonal system with the richest interval vector imaginable (H1) and the tonal system with the poorest one (H4).[48] Those tonal systems coincidently consist of the most iconic mode featuring axial symmetry – the diatonic one – and the most iconic mode featuring both axial and rotational symmetry – the whole-tone one. The fact that the latter combines both kinds of symmetry makes it the perfect bridge from this system of modes (featuring axial symmetry) to Messiaen's modes of limited transposition (featuring rotational symmetry).

At first sight, it might seem like the interval vectors of the four collections are dividing them into two groups. Just look how H1 and H2's vectors resemble each other, and so do H3 and H4's vectors. Indeed, while 3 interval classes of H1's vector coincide with H2's, and 3 interval classes of H3's vector coincide with H4's, only one interval class links H2 and H3's vectors.

However, it probably is more enlightening if, instead of comparing interval *vectors*, one attends to the individual dynamic of each interval *class*, as it progresses in the circle of modes from H1 to H4.

Interval class / Collection	minor 2nd	major 2nd	minor 3rd	major 3rd	perfect 4th	tritone
H1	2	5	4	3	6	1
H2	2	5	4	4	4	2
H3	2	6	2	6	2	3
H4	0	6	0	6	0	3

Figure 4.5

As one can see, as one pushes through the collections, from H1 till H4, each interval class changes according to a distinct dynamic of its own – the minor 2nds vanish suddenly from H3 to H4; the major 2nds have a slight increase; the minor 3rds disappear not so suddenly; the major 3rds increase; the perfect 4ths vanish even more gradually; and the tritones increase.

Which is to say that watching the interval vectors progressing from H1 to H4 is to witness the element of intervallic variety, so characteristic of H1, gradually crumbling step by step into uniformity, metamorphosing into the unambivalent H4.

Three interval classes of H1's vector, namely minor 2nds, minor 3rds and perfect 4ths, disappear completely, replaced instead by more major 2nds, major 3rds and tritones. Particularly interesting is the case of the perfect 4ths, whose aggregate decreases each time by exactly two 4ths.

Spectra variation

In 2005, Richard Krantz and Jack Douthett[49] established a method to evaluate how evenly distributed is any set of tones through the octave, by organizing them on a chromatic circle and coding their "spectra".

They first established the concepts of *specific* and *generic* length between tones. Specific length is the number of semitones between two tones. Generic length is the number of steps between two tones belonging to any scale.

For example, in C Major, the *specific* interval between C and F is 5, while the *generic* interval is 3. The generic interval is contingent on the scale, while the specific interval is not.

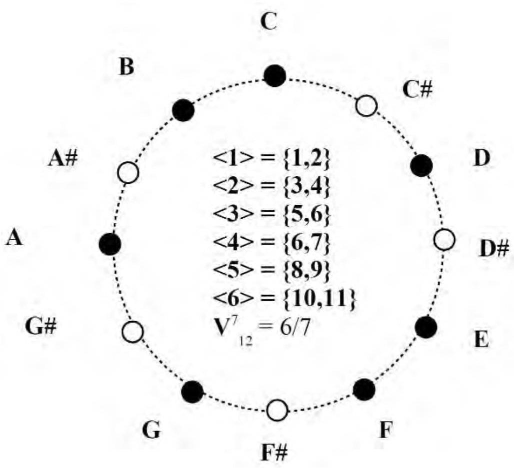


Figure 4.6 – taken from Krantz & Douthett's How Even is Even?, p.6.

The *spectra* of C Major is shown inside its distribution on the chromatic circle in *Figure 4.6*. I let Ring explain how it works:

> *"The number in angle brackets is the generic interval, i.e. we are asking "for notes that are this many steps away in the scale". The numbers in curly brackets are the*

specific intervals we find present for those steps, i.e. "between those steps we find notes that are this many semitones apart"."[50]

E.g., <1> = {1,2} means that, in C Major, notes at a distance of 1 step from each other are separated by either 1 or 2 semitones. While <2> = {3, 4} means that notes at a scalar distance of 2 steps are separated either by 3 or 4 semitones; and so on.

Whenever a spectrum's generic interval has more than 1 specific interval, one assesses its *width* by calculating the difference between the smallest and the biggest specific interval. E.g, in C Major, the width of <2> = {3, 4} consists of 4 - 3 = 1. If one sums up all the widths of a given spectra and divides it by the number of tones, we get its *variation*.

Now, the spectra variation of any scale allows one to assess how evenly distributed it is throughout the octave. Among all scales sharing a specific number of tones, there is only one whose spectra variation is less than 1 – that scale is said to have *maximally even* distribution.

By this point it should not come as a surprise that in the realm of the heptatonic collections (59 in total, according to Zeitler)[51] the maximally even set is the diatonic one. In fact, *Figure 4.6* points out already that its variation is 6/7 (that is, 0.857), with the mathematical expression $V_{12}^{7} = 6/7$.

But which collections are next in line in the quest to fill the podium of the most even "7-out-of-12"[52] sets?

The second most even heptatonic set is what Krantz and Douthett call the ascending melodic minor scale (our H2).

The third place is shared by two different collections – one is that of the harmonic minor scale, and the other is what the aforementioned authors call the "whole-tone-plus-one" scale, but I simply call H3.

Figure 4.7 shows exactly that – (a) displays the spectra of H2, whose variation is 8/7, that is, 1.143 (note, slightly higher than 1); while (b) and (c) respectively display the spectra of the harmonic minor scale and H3, both of which have a variation of 10/7, that is, 1.429.

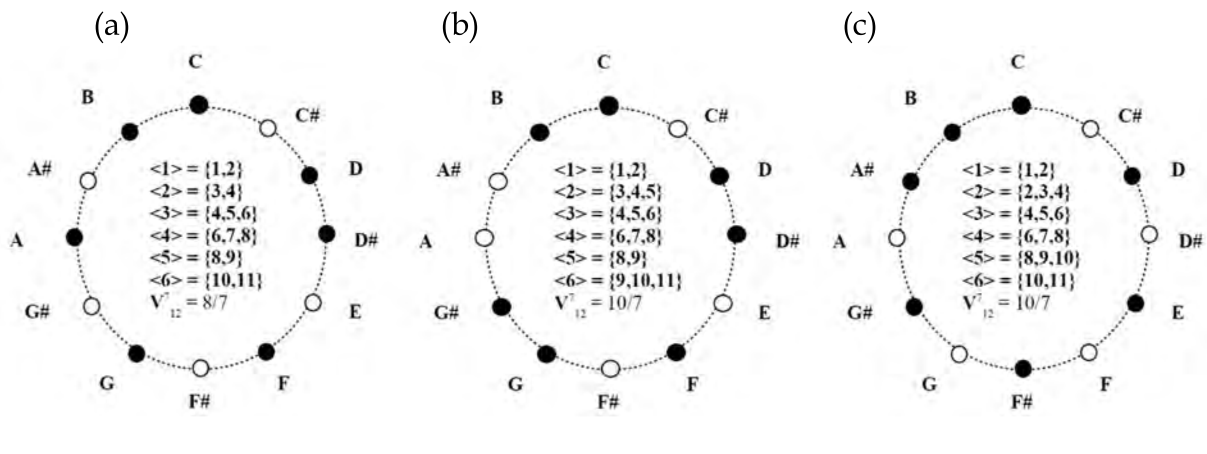


Figure 4.7 – taken from Krantz & Douthett's How Even is Even?, p.7.

What about H4, the fourth collection of our system of modes?

That scale is not mentioned in Krantz and Douthett's paper, but is just as special – among the realm of the hexatonic collections, H4 is the only one whose variation is smaller than 1. In fact, its variation is 0/6 – exactly 0! Because of that, *maximally even* distribution is not enough to define it – H4 is said to have *exactly equal* distribution.[53] *Figure 4.8* is my own version of the spectra and variation of the whole-tone scale.

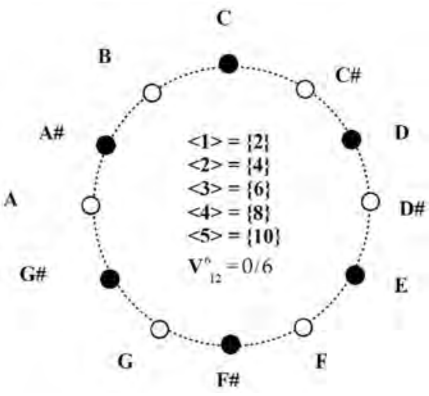


Figure 4.8 – (design inspired by Krantz & Douthett).

So, to conclude, what exactly did this property teach us about our circle of modes (as presented by the axial index)?

It made clear that the four collections making up our system of modes, whether heptatonic or hexatonic, are nothing less than the *most evenly distributed of their kind*.

Rothenberg property

A quarter of century before Krantz and Douthett published their paper, David Rothenberg was already wandering around the same field when he came up with a new model to explain the perception of musical patterns.[54] He claimed that, given the limitations of human memory, "mental reference frames" are employed to make sense of musical relations such as pitch intervals, in which case the "frames" refer to musical scales.

Rothenberg argued for dispelling the notion that "musical "intervals" are chosen for specific acoustical properties (usually their frequencies form a simple ratio) and musical scales are formed by their superimposition"[55] by pointing out that many cultures other than the European one use scales that do not reflect such a theory, and he gives the example of the Javanese music, "which uses musical intervals comprised of tones with irrational frequency ratios not approximating those to be found low in the overtone series".[56]

Instead, he gives a fascinating (even if disputable) new model to predict which scales provide that "mental reference frame" that we humans need to make sense of pitch constructions such as melodic phrases or chords.[57] He then divides any conceivable scale into three categories (and here we'll recycle from Krantz and Douthett a couple of terms regarding intervals, even if they were unbeknown to Rothenberg).

1. A scale is *strictly proper* if any *specific* interval it contains is exclusive to one of its *generic* intervals. That is to say, if generic interval *x* is larger than *y*, *x*'s specific interval will be larger than *y*'s as well. E.g., a generic 4th of a strictly proper scale will always be larger than a generic 3rd.
2. A scale is *proper*, but not *strictly* so, when it is assured that, if generic interval *x* is larger than *y*, *x*'s specific interval will be either larger or the same as *y*'s. E.g.,

    a generic 4th of a proper scale will be either larger or cover the same semitones as a generic 3rd.

3. A scale is *improper* if none of the criteria established above is assured, and the generic intervals overlap each other. E.g., a generic 4th of an improper scale might be larger, the same, or even smaller than a generic 3rd.

Rothenberg argues that both proper and strictly proper scales, regardless of the tuning system they are based upon, are the best poised to facilitate the identification of its different degrees. Not surprisingly, there are not that many of them.

From the 228 modal families identified by Ring, only 12 are strictly proper,[58] among which stand out the prestigious families of the pentatonic, the octatonic and the chromatic scales. Again according to Ring, 27 modal families are proper.[59]

But where do our three Heptatoniae and one Hexatonia fall in this honourable list?

H4, the whole-tone collection, rightly claims a spot in the exclusive list of 12 strictly proper scales.

H1, H2 and H3 all consist of proper scales, and there is only one other heptatonic scale that can claim to be so (the harmonic minor).

One can confirm these claims by going back to the spectra variations presented on the last subchapter. In the case of H4, *Figure 4.8* demonstrates that each specific interval is described by one generic interval, making it strictly proper. *Figure 4.6* exposes why isn't the H1 strictly proper as well – because one specific interval {6} is shared by two generic intervals, <3> and <4>; that is to say, because there is in the diatonic scale one generic 5th which is as large as a generic 4th (the much vilified tritone). To check that H2 and H3 are proper scales as well, check the spectra variations (a) and (c) of *Figure 4.7*.

Now, once again, we ask, what exactly did this property teach us about our circle of modes (as presented by the axial index)?

It highlighted the fact that, beyond being the most evenly distributed scales, our three Heptatoniae and one Hexatonia consist of the *most proper collections of their kind*, according to Rothenberg, as regards the creation of "mental reference frames" indispensable for our understanding of basic pitch constructions such as melodies and chords.

And, once again, H1, H2 and H3 share a distinction with the harmonic minor collection (that being, this time, to be a proper scale).

In the case of H4, it shares the quality of being the only strictly proper hexatonic scale with a curious collection, unnoticed until now by this study (but, curiously enough, not by me as a composer), which I call the half-tone/tone-and-a-half scale, similar in principle to the octatonic tone/half-tone scale.

Myhill property

This property, named after mathematician John Myhill, describes any scale which has exactly two specific intervals for every generic interval.[60] According to Ring,[61] there are only 6 modal families which feature such a property, the most relevant being the diatonic and the pentatonic collections. Remarkably, contingent on this property are a few other attributes such as "cardinality equals variety", "structure implies multiplicity" and "well-formedness", which highlight other interesting aspects of the construction of those collections.[62]

If we go back to *Figure 4.6* we can verify that, in the variation spectra of H1, each generic interval is indeed ascribed two specific intervals. However, that is the only collection of our modal system which features such a property – some of H2 and H3's generic intervals have more than two specific intervals, while each of H4's generic intervals has only one specific interval.

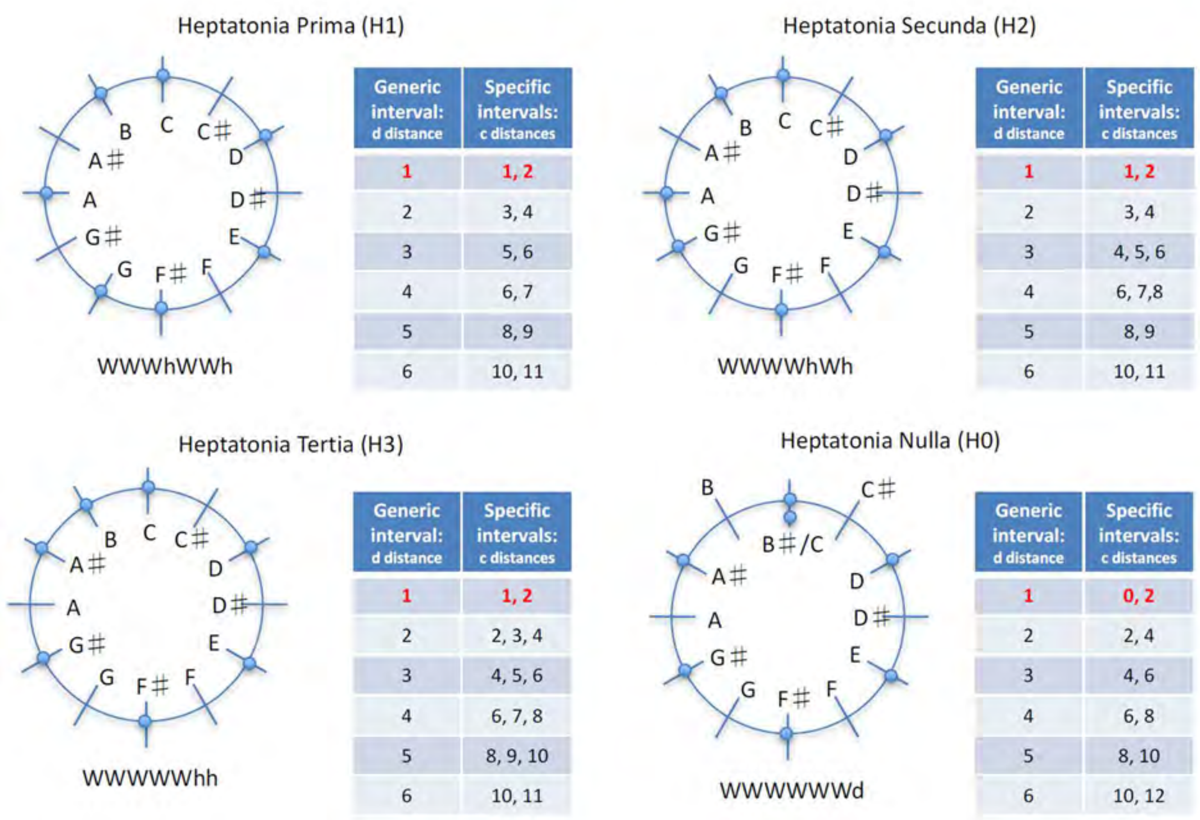
One could therefore conclude that, in contrast with the other properties discussed here, the Myhill property does not attest to anything particularly interesting in our modal system. But that would ignore the fact that, among all the attributes of the diatonic collection, the Myhill property is by far the one which better captures the essence of our modal system, if only we were to consider a "less strict" version of it.

That is precisely what Mike Hall[63] did just last year when he distinguishes two versions of that property:

1. the Strong Myhill Property (SMP) consists but of a relabel of the former. That is, concerns the scales which have two specific intervals assigned to each of their generic intervals.
2. The Weak Myhill Property (WMP) concerns the scales which have two specific intervals assigned to its "smallest non-zero generic intervals",[64] that is, the smallest scalar step – that of a generic second. This property differs from SMP because it does not take into consideration any other generic interval beyond the generic second.

What Hall does here is to extend a well-known property of the diatonic collection to encompass any scale whose consecutive steps are separated by two different intervals. He then picks those which are separated by minor and major seconds, thereby arriving to H1, H2 and H3. Nothing new here – Otterström and Bárdos had already identified those scales long before – but Hall goes two steps further.

First, he identifies a fourth scale which satisfies both SMP and WMP, which he labels Heptatonia Nulla (H0) and deems "closely related to the whole tone scale with a strategic spelling and enharmonic repeat". In fact, that scale perfectly matches the one proposed by the axial index of our modal system to fill the A*b*/G# spot – that is, the whole tone scale with one of its tones repeated enharmonically, which we reduced to its more practical hexatonic form and labelled H4, instead of H0.



Heptatonia Prima (H1)

Generic interval: d distance	Specific intervals: c distances
1	1, 2
2	3, 4
3	5, 6
4	6, 7
5	8, 9
6	10, 11

Heptatonia Secunda (H2)

Generic interval: d distance	Specific intervals: c distances
1	1, 2
2	3, 4
3	4, 5, 6
4	6, 7,8
5	8, 9
6	10, 11

Heptatonia Tertia (H3)

Generic interval: d distance	Specific intervals: c distances
1	1, 2
2	2, 3, 4
3	4, 5, 6
4	6, 7, 8
5	8, 9, 10
6	10, 11

Heptatonia Nulla (H0)

Generic interval: d distance	Specific intervals: c distances
1	0, 2
2	2, 4
3	4, 6
4	6, 8
5	8, 10
6	10, 12

Figure 4.9 – taken from Hall's The Myhill Property: the Strong vs. the Weak, Figure 2: Weak Myhill Property (WMP).

Most interestingly, he envisions that heptatonic form of the whole-tone scale as belonging to a system, which he calls "Circle of Heptatonic Scales". I let him explain:

> *"Even though it is impossible to transpose any scale within H2 or H3 by altering a single pitch chromatically as their key structures are more complex than H1, it may be surprising, however, that it is possible to transpose these scales systematically through all heptatonic scales (H0, H1, H2, and H3) taken together as a mathematical group by raising only one pitch at a time by a half step. This Circle (a cyclic group in mathematics) is a consequence of WMP."*[65]

Of course, the possibility of going through all those four scales with only one alteration at a time should not be surprising in the least to anyone familiarized with this study. However, the Circle of Heptatonic Scales which Hall suggest differs as it is based on a very specific process of modal alteration – that of the *systematic raising by half step of every degree of a scale in an orderly fashion*.

The result of this process can be better grasped by *Figure 4.10*. Notice how each degree of the initial scale is consecutively raised one half step (starting with D*b* – D, and ending with B – B#) and how each of our four collections is unveiled through that process. Now, this process is never-ending because, as Hall tells us, if we raise the first degree of H0 by a half step, we will trigger a whole new circle – the same one but raised by a semitone, that is, rooted in C#.

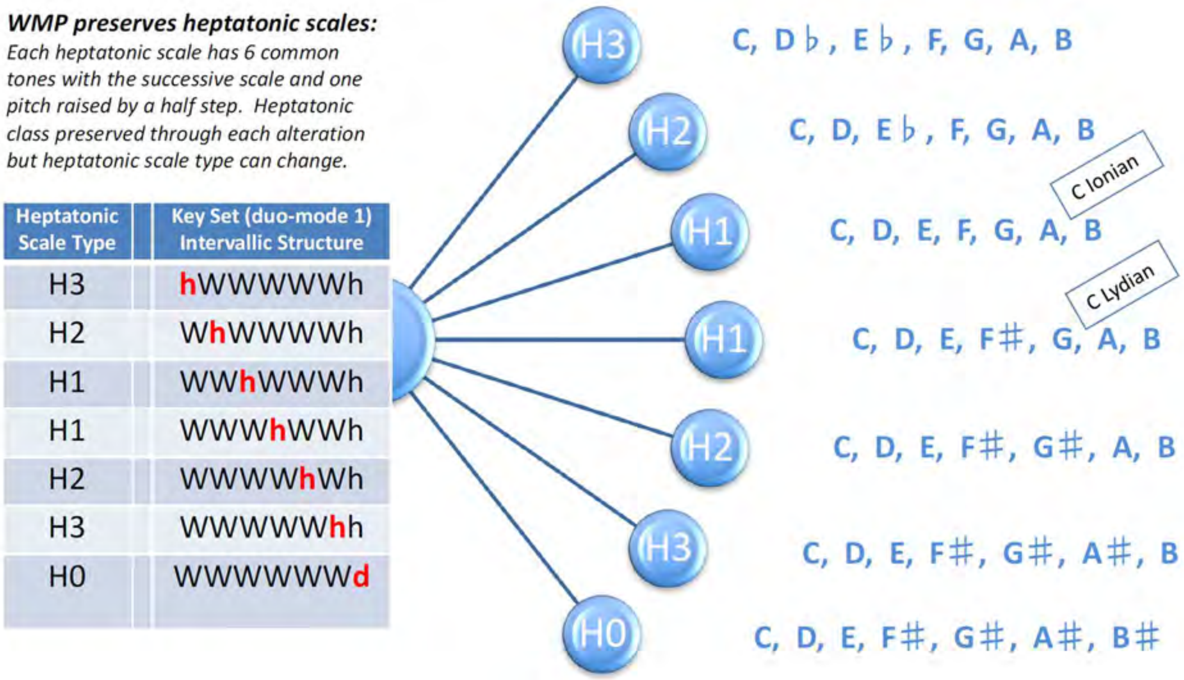


Figure 4.10 – taken from Hall's The Myhill Property: the Strong vs. the Weak, Figure 3: Constructing the Circle of Heptatonic Scales.

What can be extracted from Hall's paper is that, just as the diatonic family can be represented by a closed heptatonic circle of modes – if only we allow a mutable root between the Locrian and Lydian modes (see page 7) – the four "Heptatoniae" can be equally represented by the same closed heptatonic circle of modes, if only we allow a mutable root to bridge H0 (or H4, in my terminology) to H3. That is, no doubt, a fascinating discovery just by itself.

Still, I cannot help but wonder why Hall did not go a step further, which would be to represent all those closed heptatonic circles in an actual circle of fifths, as we did in *Figure 4.4*. In *Figure 4.11* the 7 modes featured in Hall's illustration are highlighted to better grasp how his definition of a "Circle of Heptatonic Scales" can be incorporated in this modal system.

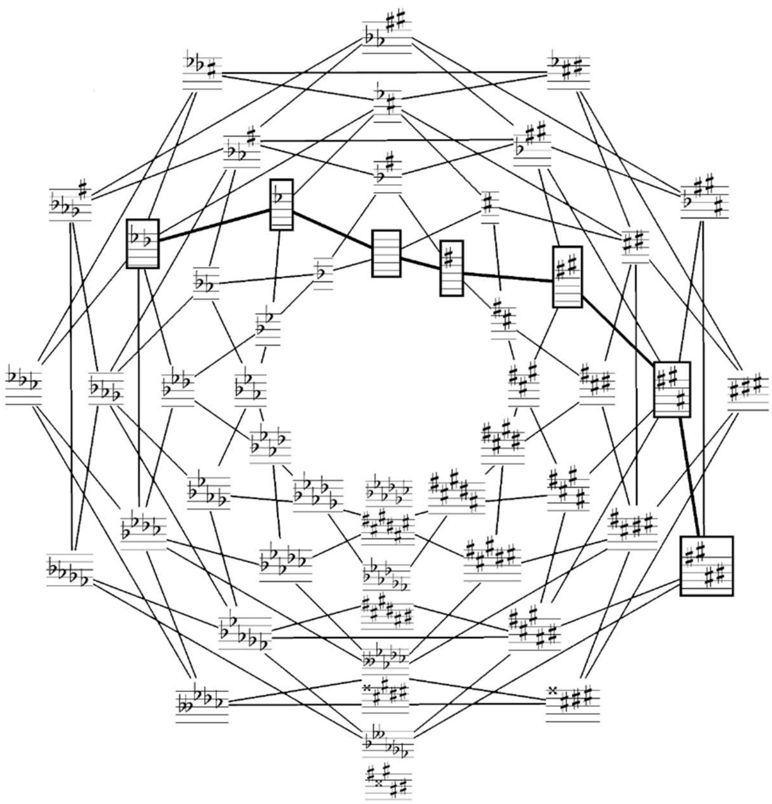


Figure 4.11

Going back to Bárdos' treaty *Heptatonia Secunda*, one learns that the idea of H2 being one step closer than H1 to the whole-tone collection was already a well-established fact among composers who did use those modes at that time. Indeed, *Figure 4.12*, taken from that treaty, emphasizes the very idea that H2's structure (represented in the middle) is the result of the fusion of the pentatonic scale with the whole-tone scale.

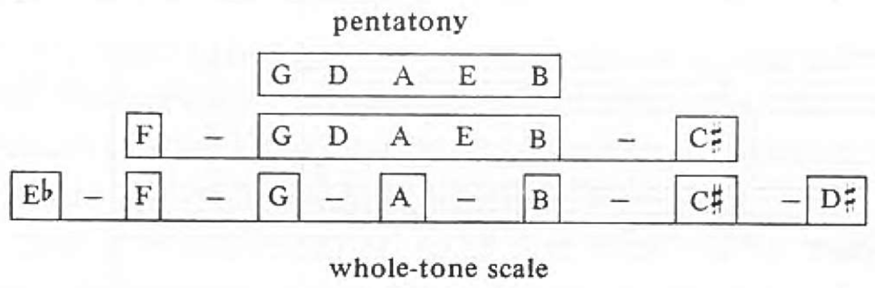


Figure 4.12 – taken from Bárdos's Selected Writings, p. 116.

However, beyond this study, Hall's paper is the only one I am aware of which claims the existence of a circle of scales incorporating the three Heptatoniae *as well as* the whole-tone scale (which, as tempting as it might be from a theoretical perspective, my composer mindset cannot bring myself to the idea of calling "Heptatonia"). Furthermore, Hall proves the link between these collections by pointing to a cyclical process of modal alteration which enables us to travel around all their transpositions. He only fails short of actually showing the resulting 48 key signatures in a circle of fifths. Conversely, he points out to the interesting possibility of this system bringing some practical benefits to the theory of music analysis, for example:

> *"Why do some chromatically altered chords support tonality (e.g., the Augmented Sixth Chords)? Italian and French Chords "naturally" belong to one of the modes of H3, so they are part of the Heptatonic Circle and support the entire structure. Augmented Sixth Chords are correctly viewed as borrowed chords under WMP."*[66]

Another closed system

What I stumbled across out of a quest for new pitch material for my compositions was to some extent unearthed through mathematical procedures by Hall, Rothenberg, Clough, and others. All things considered – the evolution of the bracelets' pattern, the dynamic of the interval vectors, the Rothenberg property, the theory of evenness, the Weak Myhill property – it looks like, regardless from which angle we look at it, the four collections extracted from the axial index form an incredibly logical and consistent modal system which deserves attention from composers and theorists alike.

But what about the prime index, and the different solution it gives for the Ab/G# spot? What will we find if, as we did for the axial index, we follow it to its logical conclusion? Is it an open system, ever generating more collections out of the three Heptatoniae, or is it a closed system parallel to the axial index?

Truth be told, once familiarized with the sheer amount of modal families one can extract from this second system, one quickly realizes a thorough exploration of the intricacies of both indexes would require a study with another order of magnitude than this one. Because of that, what follows is just a brief overview of its origin and morphology.

Let us recycle the three bracelets (a, b, c) with which we started this chapter. They represent the three Heptatoniae and we used them to guess the limits of the axial index. How is it possible to, based on the same bracelets, have a different result for bracelet (d), and from that go as far as possible on the prime index?

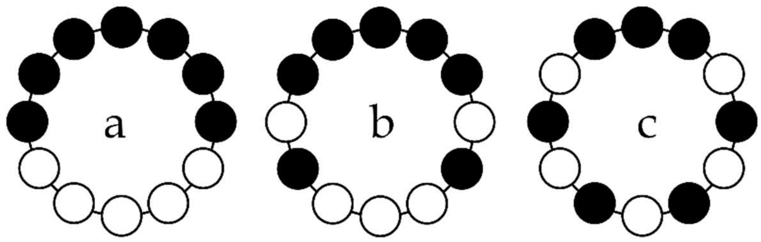


Figure 4.13

For starters, we might attribute right away a letter representing a note to each bead.

In the case of the axial index, we interpreted the different patterns as being the result of each black bead travelling downwards whenever possible. But there is also another interpretation in which the only black beads that move around – let us call them the *wanderers* – are the ones which in bracelet (a) fill the F and B spots (see *Figure 4.14*). These wanderers forever travel around the circle of fifths – the B bead goes each time a perfect 5th upwards, while the F bead goes a perfect 5th downwards. In the meantime, the other black beads stay forever still in their initial positions, which are the diatonic white keys – let us call them the *remainders* –only momentarily disappearing whenever they share the same pitch letter as a wanderer. E.g., in bracelet (c) the remainders C and E disappeared because they shared the pitch letter with the wanderers E*b* and C#. Note that sharing a letter does not mean sharing the pitch class! E.g., if one of the wanderers is a B*bb*, it will share the bead with the remainer A, but the remainer B will be the one which will momentarily disappear.

Figure 4.14 displays the patterns of the first four bracelets; this time we distinguished the wanderers (which have a white core) from the remainers (which are totally black).

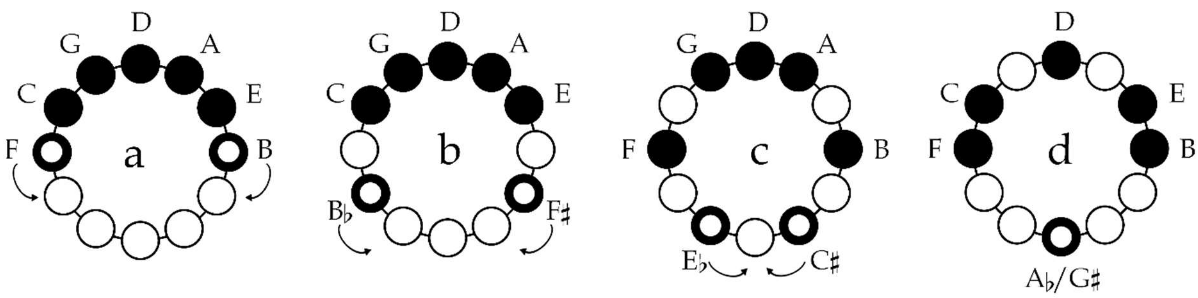


Figure 4.14

The collection represented by bracelet (d) is indeed the collection suggested by the prime index for the Ab/G# spot (see *Figure 2.7* on page 13). But this process is far from ending here. Indeed the wanderer that travels a 5th upwards might as well continue beyond bracelet (d) to go from G# to D#, A#, E#, B#, F*x*, C*x*, etc. At the same time, the wanderer which travels a 5th downwards goes from A*b* to D*b*, G*b*, C*b*, F*b*, B*bb*, E*bb*, etc.

As we can see in *Figures 4.14* and *4.15*, the patterns of the first eight collections of this vast system are already quite heterogeneous – we have pentatonic collections (g, h), as well as hexatonic (d), heptatonic (a, b, c, f), and octatonic ones (e).

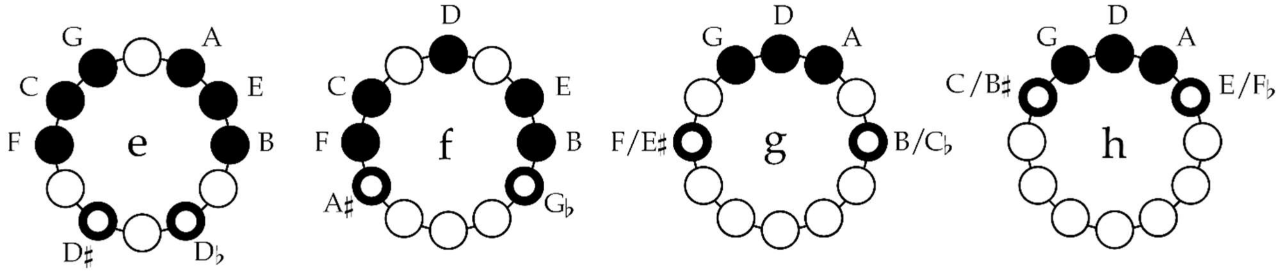


Figure 4.15

And yet, these 8 bracelets are not but a small peek into the vast modal system which lingers behind the prime index, and that begs the question – does this system even end? Can't the wanderers after all go round and round the circle of fifths forever, generating ever more pitch material?

Actually, this system is as closed as the one based on the axial index. Indeed, the process of altering a mode following both directions of the circle of fifths – a process which I denominate of *reflected modal alteration* – has a finite number of iterations (84, to be precise) until it arrives precisely to where it started, to bracelet (a).[67] And yet, notwithstanding the 84 repetitions, most of the resulting patterns are reiterations of the same collections – one can only extract 17 different ones from this system, 8 of which are already represented by bracelets (a) to (h).

These collections, as distinct as they are from each other, have at least one feature in common – they are all palindromic (even if in some cases the axis of reflection falls *between* tones).

It is also worth mentioning that, even if the link between their key signatures is not as self-evident and crystalline as the one bonding the Heptatoniae, the structures of these scales are very rewarding from a composer's perspective, as they fundamentally consist of variations or ornamentations around a few of the most elemental divisions of the octave.

Epilogue

Having arrived to the end of this thesis, it is well worth reflecting for a moment about how the modal system it slowly brought to existence, whether in the form of a triple circle (see chapter 3) or a quadruple one (see chapter 4), might be applied by composers as a tool for the harmonic or melodic foundations of their pieces.

Of course, it only takes a quick peek into the History of Western Music to dispel any doubts about the natural proclivity of each of this system's collections to provide powerful compositional tools, such as grids of harmonic functions (essential for the creation of a tonal language), and basic structures for melodic and harmonic elaboration (the "mental reference frames" discussed by Rothenberg).

After all, H1 (the diatonic collection) is almost as old as Western Music's harmonic thinking itself, and has been ubiquitous ever since its formulation.

To the same effect, H2 traditionally served an important role in Hungarian musical conscience (according to Bárdos).[68] And its influence does not stop there; in fact, its modes can be traced everywhere – in Grieg and Sibelius' piano pieces, in Jazz standards, even in the main theme of the TV series *The Simpsons*.

H3 is by far the most obscure of our collections, having been explored in a somewhat consistent way by Bartók and few others. Though we can understand the reasons behind its historical neglect (having to do with the scale's weird, "whole-tonish" structure), that shouldn't blind us to the other side of the coin, which is the opportunity to ground our harmonic thinking on fresh terrain.

And finally, H4 (the whole-tone collection) – introduced to the Western musical scene by Russian composers at the end of the 19th century, and consolidated as a crucial part of its idiom by French composers at the beginning of the 20th – requires neither introduction nor further eulogising.

Still, it is not the individual merit of the aforementioned collections that I'm putting under a composer's scrutiny here, but the system as a whole that brought them together. Indeed, I could end this thesis reminding the reader of all the properties attesting to the bond between these four collections; or conjuring once again the image of their 48 keys organically linked by proximity while evenly distributed in space. But instead, as a prophylactic measure aimed at any aspiring fellow composer who might get some ideas from this thesis, I prefer to put forth a couple of thoughts for his or her consideration:

First one – if I ever gave the impression that the four collections are somehow divided between the three Heptatoniae and the Hexatonia, or, on the contrary, made it seem like the quadruple circle of modes displays its collections in perfect continuity, the practical application of the system provides us with a stark reality check – it is not as much a spectrum as it is a bipolar system with two areas of influence.

The "rival" collections are of course, H1 and H4. Implied by H1 is the pentatonic collection (its negative) and by H4 the other Messiaen's modes of limited transposition. Regarding the two areas of influence, H2 clearly sides with H1, while there is no doubt about H3's subservience to H4. That becomes clear if one checks the rift between H2 and H3's interval vectors (see page 36) but, more fundamentally, that's how these collections are paired when attending to their aural recognition. It is no coincidence that H2 is widely regarded – by musicologists referring to one of its modes (the ascending melodic minor) and by Jazz musicians alike – as part of the diatonic realm; and that H3 is invariably classified with a reference to H4 – be it "whole-tone-plus-one" scale,[69] or "leading whole-tone" scale.[70]

This duality between diatonic and "whole-tonic" could be well worth exploring further – by applying its knowledge in the analyses of works by Debussy and Bartók, for example; or by directly informing our own compositions through a new way of thinking these "tonal-idioms". And that leads me to the second point.

Theory is abstract, while a composition only materializes when it is both performed and listened to (even if only by the performers themselves); therefore, writing a piece with no thought regarding how it will be listened to is, to an extent, as absurd and far-fetched an idea as writing a piece with no thought regarding how it will be performed.

Once a pitch set emancipates from the system behind its inception and enters a composition, it stops existing in a vacuum; from that moment, it will be comprehended and judged according to its integration in the structure of the piece, and both by the natural predispositions of the human ear and by the awareness of its historical context. Now, I'm not qualified to address the psychology or mechanics of human aural reception (although if I had to guess, I'd say some of the properties addressed in Chapter 4, such as Rothenberg's, could be key to understanding some of our aural predilections). However, as far as the historical context is concerned, our new circle of modes could not be any clearer.

As I mentioned at the beginning of the epilogue, the fact is that the four collections which constitute our system are all over the map as regards their historical connotations. Furthermore, not only do they point to distinct musical traditions, but some of them are inconsistent even at the individual modal level. This poses a huge challenge for any composer willing to make full use of what this system has to offer while at the same time being congruent in the selection of his or her pitch structures.

Because no matter how admirable and wholesome one might deem the internal logic of a system of pitch sets or a technique of pitch organization (or, for that matter, any theory about any parameter of music), one will still have to assess the cultural and historical baggage of its material if he or she is to successfully apply it in a composition.

From the beginning of my investigation I was equally drawn to the new circle of modes from both theoretical and compositional standpoints. The theory, I believed, would give me the opportunity to practice rational enquiry and to exercise the more methodical aspect of my nature, while the composition would give me the opportunity to exercise artistic freedom grounded on a solid modal framework. But whilst I cherish each one of this system's individual collections and I find them very useful for setting in motion the process of generating pitch material, what comes afterwards is what really defines a piece – the process of developing that same material. And I have never felt comfortable with any constraint, regardless of which system or theory it comes from, to the freedom of moulding the piece's harmonic identity as I wish, even if that takes me to places the system would deem “uncalled-for” (such as pitch structures with 3 or 4 consecutive half steps, which the new circle simply does not cover…).

What that translates in my music is that one rarely finds any of the four collections in its pure form for long stretches of time. Instead, one will most often find: (a), modes which are the outcome of the superimposition of two collections; (b), modes taken from the extended prime index, prompted by the technique of reflected modal alteration,[71] and which have a whole logic of their own that I find weirder and more compelling; and (c), free modal development generated out of one of the collections and based on melodic contour.[72]

And yet, it does not follow that I personally have no practical use for the new circle of modes when it stands in its pure, unadulterated form. In fact, one thing in particular I intend to explore moving forward is the potential of the axial index (see *Figure 2.11*) to become a useful compositional tool for distinguishing between different grades of brightness in harmonic structures. Testing that requires only to play through the 12 modes uninterruptedly, starting on the brightest one somewhere on the circle’s top-left quadrant (which one exactly depends on the point of view) and going clockwise to the darkest mode (in the opposite quadrant) and back to the brightest.

So smooth is that process (in spite of the fact that it requires six changes of modal family) that I also think the quadruple circle of modes might prove to be an interesting didactic tool to help minds propel their musical curiosity out of the diatonic world towards more remote, adventurous harmonic universes, and all that without even noticing the journey.

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Notes

Introduction

[1] Keith, M. (1991), *From Polychords to Pólya: Adventures in Musical Combinatorics*, Vinculum Press, p. 90.

[2] The reason for the disparity being the assumptions Zeitler has of a scale. "A scale being defined as a series of notes, each of higher pitch than the previous, the entire series of pitches less than an octave (so the sequence can repeat itself at each octave), where the step between successive notes in the scale is no greater than a major third. The requirement of no step greater than a major third is somewhat arbitrary, but does allow us to include commonly used scales like the Pentatonic. Allowing steps greater than a major third significantly increases the number of possible scales." Zeitler, W., *All the Scales*, viewed 3 June 2020, < https://allthescales.org/index.php>.

[3] According to John Vincent, writing in the 1950s, a scale which links "the τόνοι of ancient Greece, the eight modes of Pope Gregory, the twelve of Glareanus, and the two used almost exclusively for the past three centuries." Vincent, J. (1951), *The Diatonic Modes in Modern Music*, University of California Press, p. 1.

[4] Once again, Vincent backs my words here when he states that, as far as Western music is concerned, "departures from the basic diatonic forms are but mutations through the use of superimposed "chromatics." These chromatics (half-tones and sometimes even smaller intervals) have always been subservient to the diatonic scales and are thus not so much smaller subdivisions of the octave as they are subdivisions of the whole-tones of the diatonic modes." Vincent, *The Diatonic Modes in Modern Music*, p. 1.

[5] Bárdos, L. (1963), *Heptatonia Secunda: a Unique Tonal System and its Modes in the Works of Zoltán Kodály*, In Bárdos, L. (1984), *Selected Writings*, Editio Musica Budapest.

Chapter 1

[6] Messiaen, O. (1944), The Technique of my Musical Language, Chapter 16, translated by Satterfield J. (1956), Alphonse Leduc, Paris.

[7] Macroharmony, as defined by Dmitri Tymoczko, refers to the "total collection of notes heard over moderate spans of musical time." Tymoczko, D. (2011), *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*, Oxford University Press, p. 4.

[8] Some of Messiaen's modes are less aurally recognizable than others.

[9] Johnson, T. (2008), *Foundations of Diatonic Theory*: *a Mathematically Based Approach to Music Fundamentals*, The Scarecrow Press, p. 20.

[10] Vincent, *The Diatonic Modes in Modern Music*, p. 1.

[11] Term coined by Ian Ring. Ring, I., *A Study of Scales*, In *The Exciting Universe of Music Theory*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#reflective>.

[12] Vincent, *The Diatonic Modes in Modern Music*, p. 20.

[13] Serre, J. A. (1753), *Essais sur les Principles de l'Harmonie*, Paris, Prault Petit-Fils, pp. 143, 144.

[14] Ziehn, B. (1912), *Canonical Studies: A New Technic in Composition*, Milwaukee, Wm. A. Kaun Music Co., p. 1.

[15] A pitch class consists of a specific pitch – e.g., C – encompassing all its octaves and enharmonic equivalents.

[16] The exception is the tritone, which indicator shall be doubled to get the correct number of common tones.

[17] Term coined by Ian Ring. Ring, I., *Mutant Modes*, In *The Exciting Universe of Music Theory*, viewed 3 June 2020, < https://ianring.com/musictheory/mutantmodes/>.

[18] Ring, *Mutant Modes*, viewed 3 June 2020, <https://ianring.com/musictheory/mutantmodes/>.

Chapter 2

[19] The Ionian/Major mode, sometimes paired with the Aeolian/Minor mode.

[20] Even if, for argument's sake, we tried to forget the fact that the traditional circle of fifths does consist of the 12-fold transposition of the same mode, and instead we choose one tone to be the root of the whole system, and we tried to extract as many modes as possible out of that root, we would still get no more than seven different modes – the inevitably diatonic ones - until they would start repeating themselves in a different root.

[21] On the nomenclature of diatonic modes, Vincent writes that "in France three systems seem to be current: the traditional Roman Catholic Church numerical designation, a pseudo-Greek terminology, and a "white-note" characterization, i.e., *mode de re*, *mode de mi*, *mode de fa*, etc." Vincent, *The Diatonic Modes in Modern Music*, p. 3.

[22] Risking stating the obvious, any system comprising twelve modes cannot possibly rest itself solely on the diatonic family, because the latter – invariably a seven tones' set – can only provide seven different modes to the former.

[23] Vincent, *The Diatonic Modes in Modern Music*, pp. 18, 19.

[24] Distance in fifths of every altered note to the axis of the diatonic space (D):
D# - 7; E*b* – 5   F# - 4; G*b* – 8   G# - 6; A*b* – 6   A# - 8; B*b* – 4   C# - 5; D*b* – 7.

[25] Still, this operation is not as clean as the other modes' alterations, as none of the modes' degrees match each other (with the obvious exception of the root tone) – when in their axial transposition (in D), C# mode's 2nd degree is E*b*, while E*b* mode's 2nd degree is E; C# mode's 3rd degree is F*b*, while E*b* mode's 3rd degree is F#, and so on.

Chapter 3

[26] One explanation for why that is the case was given already on page 14.

[27] The Major and Minor ones.

[28] Johnson, *Foundations of Diatonic Theory*: *a Mathematically Based Approach to Music Fundamentals*, p. 20.

[29] "Just as it is useful to simplify music to pitches (disregarding timbre, dynamics, etc.) and to simplify pitches to pitch-classes (disregarding octave distinctions), so is it useful to simplify ordered successions of pc to *set*, or unordered collections of pc (disregarding distinctions of "which came first")." Rahn, J. (1980), *Basic Atonal Theory*, Schirmer Books, p. 27.

[30] To prove this, one only needs to count the intervals of any of those three collections starting on D. Clockwise or anticlockwise, the result will be the same.

[31] In this case, that means: $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

[32] The reason being the same I gave for the existence of an axial index in the first place, and I quote: «for the purpose of evaluating the proximity between a plurality of modes, it is easier to assess their intervallic differences if they all are transposed to the same root tone – this way, one needs only to spot the differences in their key signatures!» (see page 14).

[33] Anonymous, *Heptatonic scale*, In *Wikipedia, the free encyclopedia*, last edited on 5 November 2019, <https://en.wikipedia.org/wiki/Heptatonic_scale#Heptatonia_tertia>.

[34] Keith, *From Polychords to Pólya: Adventures in Musical Combinatorics*, p. 94.

[35] Drabkin, W. (2001), *Diatonic* entry, In *Grove Music Online*, Oxford University Press, viewed 5 June 2020,
<https://www.oxfordmusiconline.com/grovemusic/view/10.1093/gmo/9781561592630.001.0001/omo-9781561592630-e-0000007727?rskey=B2Lyxv&result=1>.

[36] Latham, A. (2011), *Diatonic* entry, In *The Oxford Companion to Music*, Oxford University Press.

[37] Johnson, *Foundations of Diatonic Theory*: *a Mathematically Based Approach to Music Fundamentals*, p. 20.

[38] Otterström, T. (1935), *A Theory of Modulation*, The University of Chicago Press.

[39] Otterström, *A Theory of Modulation*, p. 132.

[40] Otterström, *A Theory of Modulation*, p. 132.

[41] Otterström, *A Theory of Modulation*, p. 134.

[42] Bárdos, *Selected Writings*, p. 185.

[43] ”Bárdos pointed out that in addition to the common diatonic system, in which the tones and semitones take up the position 2-1-3-1 (i.e. a 1 + 1 1/2 + 1 + 1 + 1 + 1/2-tone set, or its modes that may be started from any tone), and which thus presents the first, most natural heptatony (let us call it heptatonia prima!), there are two further heptatonic systems, heptatonia secunda and heptatonia tertia, offering the same number of modes and transposition possibilities.” Somfai, L. (1977), *Strategics of Variation in the Second Movement of Bartók's Violin Concerto 1937-1938*, In *Studia Musicologica Academiae Scientiarum Hungaricae, T. 19, Fasc. 1/4*, Akadémiai Kiadó, p. 169.

[44] ”Did the composer *consciously* use this tonal-system with its seven modes and with the many different tonalities of each of these latter? We think not. To great creative minds, it is granted that their thoughts ”just simply come” without any premeditation. It is certain, however, that, in most creative work, awareness carries inspiration through to completion.” Bárdos, *Selected Writings*, p. 215.

Chapter 4

[45] The original bracelet diagrams, the ones conceived by Ring, differ as their tones are sequenced chromatically, and not by fifths.

[46] Gamer, C. (1967), *Deep Scales and Difference Sets in Equal-Tempered Systems*, American Society of University Composers: Proceedings of the Second Annual Conference, pp. 113–122.
Although ”Gamer attributed his contributions on deep scales to an unpublished paper by Terry Winograd (“An Analysis of the Properties of ‘Deep Scales’ in a T-Tone System,” unpublished, n.d.).” Johnson, *Foundations of Diatonic Theory*: *a Mathematically Based Approach to Music Fundamentals*, p. 157.

[47] Ring, *A Study of Scales*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#deepscale>.

[48] That H4 has one of the poorest interval vectors imaginable becomes even clearer if one considers the fact that the indicator of tritones – 3 – could just as well be doubled (given the nature of the interval) to become 6, the same value as the indicator of major seconds and major thirds.

[49] Krantz, R. & Douthett, J. (2005), *Circular Distributions and Spectra Variations in Music: How Even is Even?*, Bridges Conference, Mathematical Connections in Art, Music, and Science.

[50] Ring, *A Study of Scales*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#evenness>.

[51] Zeitler, *All the Scales*, viewed 3 June 2020, <https://allthescales.org/index.php>.

[52] Krantz & Douthett, *Circular Distributions and Spectra Variations in Music: How Even is Even?*, p. 7.

[53] Krantz & Douthett, *Circular Distributions and Spectra Variations in Music: How Even is Even?*, p. 8.

[54] Rothenberg, D. (1978), *A Model for Pattern Perception with Musical Applications Part I: Pitch structures as order-preserving maps*, Mathematical Systems Theory 11

[55] Rothenberg, *A Model for Pattern Perception with Musical Applications Part I*, p. 2.

[56] Rothenberg, *A Model for Pattern Perception with Musical Applications Part I*, p. 3.

[57] Truth be said, Rothenberg's ambitions go much further than the elaboration of a theory that concerns only the different degrees of perception of a musical scale; admittedly, his model is to be applied to topics as disparate as "musical timbres", "phonemes of a spoken language", and even "certain aspects of visual perception". However, given the subject-matter of this study, we will concentrate on its application on musical scales.

[58] Ring, *A Study of Scales*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#propriety>.

[59] Ring, *A Study of Scales*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#propriety>.

[60] Clough, J., Engebretsen, N., Kochavi, J. (1999), *Scales, Sets, and Interval Cycles: a Taxonomy*, Music Theory Spectrum 21, p. 76.

[61] Ring, *A Study of Scales*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/#myhill>.

[62] I cannot cover all those properties in this study. To learn more, check Timothy Johnson's *Foundations of Diatonic Theory, A Mathematically Based Approach to Music Fundament*, The Scarecrow Press (2008).

[63] Hall, M. (2019), *The Myhill Property: the Strong vs. the Weak, Group-Theoretic Structures among 7-Tone Equal Tempered Scales*, Society for Music Theory conference.

[64] Hall, *The Myhill Property: the Strong vs. the Weak*, p. 1.

[65] Hall, *The Myhill Property: the Strong vs. the Weak*, p. 2.

[66] Hall, M. (2019), *The Myhill Property: the Strong vs. the Weak*, Presentation.

[67] With a caveat – two of its notes are displaced an octave away from the normal disposition of a scale's tones.

Epilogue

[68] Phrases like the following one, claiming some kind of Hungarian predilection for the H2's tonal system rather than H1's, pop every so often in Bárdos' treaty. "Despite the traditional A-major key-signature: *f* and *g* even in the final cadences of a piece! The contrasting, "western" *g#-a* leading-tone movement strikes us as something of a later development – and the melody which does not readily accept the leading-tone therefore much more "fitting for Hungarians"." Bárdos, *Selected Writings*, p. 102.

[69] Krantz & Douthett, *Circular Distributions and Spectra Variations in Music: How Even is Even?*, p. 7.

[70] Ring, I., *Amazing Scale Finder*, In *The Exciting Universe of Music Theory*, viewed 3 June 2020, <https://ianring.com/musictheory/scales/3413>.

[71] See page 44 - *Another closed system*.

[72] As defined by Lasse Thoresen. See Thoresen, L. (2015), *Contour Analysis*, In *Emergent Musical Forms: Aural Explorations*, Studies in Music from the University of Western Ontario, pp. 340-343.